REGULARITY OF THE GEODESIC EQUATION IN THE SPACE OF SASAKIAN METRICS

PENGFEI GUAN AND XI ZHANG

1. Introduction

In this paper, we treat a complex Monge-Ampère type equation arsing from Sasakian geometry. There is a renewed interest on Sasakian manifolds recently, as Sasakian manifolds provide rich source of constructing new Einstein manifolds in odd dimensions [2] and its important role in the superstring theory in mathematical physics [23, 24]. Here we devote to the regularity analysis of a geodesic equation in the space of Sasakian metrics $\mathcal{H}$ (definition in (1.2)) and some of geometric applications. This equation was introduced in [19]. We believe it encodes important geometric information. This geodesic approach is modeled in Kähler case [22, 29, 10, 5, 4, 26]. The C_w^2 (see definition 1) regularity proved by Chen [5] for the geodesic equation in the space of Kähler metrics has significant geometric consequences. We will deduce the parallel results in Sasakian geometry.

A Sasakian manifold (M, g) is a $2n + 1$-dimensional Riemannian manifold with the property that the cone manifold $(C(M), \tilde{g}) = (M \times \mathbb{R}^+, r^2 g + dr^2)$ is Kähler. A Sasakian structure on M consists of a Reeb field ξ of unit length on M, a $(1, 1)$ type tensor field $\Phi(X) = \nabla_X \xi$ and a contact 1-form η (which is the dual 1-form of ξ with respect to g). (ξ, η, Φ, g). Φ defines a complex structure on the contact sub-bundle $\mathcal{D} = ker\{\eta\}$. $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ provides M a transverse Kähler structure with Kähler form $\frac{1}{2}d\eta$ and metric g^T defined by $g^T(\cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \Phi\cdot)$. The complexification $\mathcal{D}^C$ of the sub-bundle $\mathcal{D}$ can be decomposed it into its eigenspaces with respect to $\Phi|_{\mathcal{D}}$ as $\mathcal{D}^C = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. A p-form θ on Sasakian manifold (M, g) is called **basic** if
$$i_\xi \theta = 0, \quad L_\xi \theta = 0$$
where i_ξ is the contraction with the Reeb field ξ, L_ξ is the Lie derivative with respect to ξ. The exterior differential preserves **basic** forms. There is a natural splitting of the complexification of the bundle of the sheaf of germs of **basic** p-forms $\wedge_B^p(M)$ on M,

$$(1.1) \qquad \wedge_B^p(M) \otimes C = \oplus_{i+j=p} \wedge_B^{i,j}(M),$$

where $\wedge_B^{i,j}(M)$ denotes the bundle of basic forms of type (i, j). Accordingly, ∂_B and $\bar{\partial}_B$ can be defined. Set $d_B^c = \frac{1}{2}\sqrt{-1}(\bar{\partial}_B - \partial_B)$ and $d_B = d|_{\wedge_B^p}$. We have $d_B = \bar{\partial}_B + \partial_B$, $d_B d_B^c = \sqrt{-1}\partial_B\bar{\partial}_B$, $d_B^2 = (d_B^c)^2 = 0$. Denote the space of all smooth **basic** real function on M by $C_B^\infty(M)$. Set

$$(1.2) \qquad \mathcal{H} = \{\varphi \in C_B^\infty(M) : \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0\},$$

where

$$(1.3) \qquad \eta_\varphi = \eta + d_B^c\varphi, \quad d\eta_\varphi = d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi.$$

The space $\mathcal{H}$ is contractible. For $\varphi \in \mathcal{H}$, $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ is also a Sasakian structure on M, where

$$(1.4) \qquad \Phi_\varphi = \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi, \quad g_\varphi = \frac{1}{2} d\eta_\varphi \circ (Id \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi.$$

$(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ and (ξ, η, Φ, g) have the same transversely holomorphic structure on $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$ (Proposition 4.2 in [14], also [1]). Conversely, if $(\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ is another Sasakian structure with the same Reeb field and the same transversely holomorphic structure on $\nu(\mathcal{F}_\xi)$, then $[d\eta]_B$ and $[d\tilde{\eta}]_B$ belong to the same cohomology class in $H_B^{1,1}(M)$. There exists a unique basic function (e.g., [12]), $\tilde{\varphi} \in \mathcal{H}$ up to a constant such that

$$(1.5) \qquad\qquad d\tilde{\eta} = d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \tilde{\varphi}.$$

If (ξ, η, Φ, g) and $(\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ induce the same holomorphic structure on the cone $C(M)$, then there must exist a unique function $\varphi \in \mathcal{H}$ up to a constant such that $\tilde{\eta} = \eta_\varphi$, $\tilde{\Phi} = \Phi_\varphi$ and $\tilde{g} = g_\varphi$. $\mathcal{H}$ encodes rich information on the Sasakian manifolds. We call $\mathcal{H}$ the space of Sasakian metrics.

Let's briefly recall the geodesic equation in $\mathcal{H}$ introduced in [19]. $d\mu_\varphi = \eta_\varphi \wedge (d\eta_\varphi)^n$ defines a measure in $\mathcal{H}$, a Weil-Peterson metric in the space $\mathcal{H}$ can be defined as

$$(1.6) \qquad\qquad (\psi_1, \psi_2)_\varphi = \int_M \psi_1 \cdot \psi_2 d\mu_\varphi, \quad \forall \psi_1, \psi_2 \in T\mathcal{H}.$$

Since the tangent space $T\mathcal{H}$ can be identified as $C_B^\infty(M)$, the corresponding geodesic equation can be expressed as

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial \varphi}{\partial t}|^2_{g_\varphi} = 0,$$

where g_φ is the Sasakian metric determined by φ. A natural connection of the metric can be deduced from the geodesic equation. In [19], we proved that this natural connection is torsion free and compatible with the metric, there is a splitting $\mathcal{H} \cong \mathcal{H}_0 \times \mathbb{R}$, $\mathcal{H}_0$ (defined in (6.5)) is totally geodesic and totally convex, the corresponding sectional curvature of $\mathcal{H}$ is non-positive.

A natural question raised in [19] is: *given two functions φ_1 and φ_2 in $\mathcal{H}$, can they be connected by a geodesic path?*

The question is equivalent to solve a Dirichlet problem for a degenerate fully nonlinear equation,

$$(1.7) \qquad \begin{cases} \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial \varphi}{\partial t}|^2_{g_\varphi} = 0, & M \times (0,1) \\ \varphi|_{t=0} = \varphi_0 \\ \varphi|_{t=1} = \varphi_1 \end{cases}$$

It was discussed in [19], when $n = 1$, equation (1.7) is related to the corresponding geodesic equation introduced by Donaldson [11] for the space of volume forms on Riemannian manifold with fixed volume [19]. Recent work of Chen and He [6] implies the existence of a C_w^2 geodesic. The main goal of this paper is to establish the existence and regularity of solutions to geodesic equation (1.7) in any dimension.

Our first step is to reduce geodesic equation (1.7) on $\mathcal{H}$ to a Dirichlet problem of complex Monge-Ampère type equation on the Kähler cone $C(M) = M \times \mathbb{R}^+$. Let $\varphi_t : M \times [0,1] \to R$ be a path in the metric space H^0. Define a function ψ on

$\overline{M} = M \times [1, \frac{3}{2}] \subset C(M)$ by translating time variable t to the radial variable r as follow,

$$(1.8) \qquad \psi(\cdot, r) = \varphi_{2(r-1)}(\cdot) + 4 \log r.$$

Setting a $(1,1)$ form on $\overline{M}$ by

$$(1.9) \qquad \Omega_\psi = \bar{\omega} + \frac{r^2}{2}\sqrt{-1}(\partial\bar{\partial}\psi - \frac{\partial\psi}{\partial r}\partial\bar{\partial}r),$$

where $\bar{\omega}$ is the fundamental form of the Kähler metric $\bar{g}$.

The key observation is that the Dirichlet problem (1.7) is equivalent to the following Dirichlet problem of a degenerate Monge-Ampére type equation

$$(1.10) \qquad \begin{cases} (\Omega_\psi)^{n+1} = 0, & M \times (1, \frac{3}{2}), \\ \psi|_{r=1} = \psi_1, \\ \psi|_{r=\frac{3}{2}} = \psi_{\frac{3}{2}} \end{cases}$$

The following proposition will be proved in section 2.

Proposition 1. *The path φ_t connects $\varphi_0, \varphi_1 \in \mathcal{H}$ and satisfies the geodesic equation (1.7) if and only if ψ satisfies equation (1.10), where ψ and Ω_ψ defined as in (1.9), $\Omega_\psi|_{D^C}$ is positive and $\psi|_{M \times \{1\}} = \varphi_0$, $\psi|_{M \times \{\frac{3}{2}\}} = \varphi_1 + 4\log\frac{3}{2}$.*

Equation (1.10) is degenerate. In order to solve it, we consider the following perturbation equation

$$(1.11) \qquad \begin{cases} (\Omega_\psi)^{n+1} = \epsilon f\bar{\omega}^{n+1}, & M \times (1, \frac{3}{2}), \\ \psi|_{r=1} = \psi_1, \\ \psi|_{r=\frac{3}{2}} = \psi_{\frac{3}{2}} \end{cases}$$

where $0 < \epsilon \le 1$ and f is a positive basic function. Also, we set the following approximate for (1.7)

$$(1.12) \qquad \begin{cases} (\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{4}|d_B\frac{\partial\varphi}{\partial t}|_{g_\varphi}^2)\eta_\varphi \wedge (d\eta_\varphi)^n = \epsilon\eta \wedge (d\eta)^n, & M \times (0,1) \\ \varphi|_{t=0} = \varphi_0 \\ \varphi|_{t=1} = \varphi_1 \end{cases}$$

Equation (1.11) is a degenerate elliptic complex Monge-Ampère type equation. The Dirichelt problem for homogeneous complex Monge-Ampère equation was initiated by Chern-Levine-Nirenberg in [8] in connection to holomorphic norms. The regularity of the Dirichlet problem of the complex Monge-Ampère equation for strongly pseudoconvex domains in $\mathbb{C}^n$ was proved by Caffarelli-Kohn-Nirenberg-Spruck in [3]. In general, $C^{1,1}$ regularity is optimal for degenerate complex Monge-Ampère equations (e.g., [16, 5, 18]). The goal is to get some uniform estimate on $\|\psi\|_{C_w^2(\overline{M})}$ for solutions of elliptic equation (1.11) independent of ϵ. Equation (1.11) differs from the standard complex Monge-Ampère equation in [31] on Kähler manifolds in a significant way, as Ω_ψ involves also the first order derivative term. The complex Monge-Ampère equations of this type also arise naturally in other contexts, for example, in superstring theory studied in Fu-Yau [13]. We believe the analysis developed in this paper for equation (1.11) will be useful to treat general type of complex Monge-Ampère equations.

Definition 1. *Denote $C_w^2(\bar{M})$ the closure of smooth function under the norm*

(1.13) $$\|\psi\|_{C_w^2(\overline{M})} = \|\psi\|_{C^1(\overline{M})} + \sup_{\overline{M}} |\Delta\psi|.$$

We say ψ is a C_w^2 solution of equation (1.10) if $\psi \in C_w^2(\bar{M})$ such that $\Omega_\psi \geq 0, \Omega_\psi^{n+1} = 0, a.e..$ For any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, we say φ is a C_w^2 geodesic segment connecting φ_0, φ_1, if ψ defined in (1.8) is a C_w^2 solution of equation (1.10).

The main result of this paper is the following a priori estimates.

Theorem 1. *For any positive basic smooth f and for any given smooth boundary data in $\mathcal{H}$, there is a unique smooth solution ψ to the equation (1.11). Moreover, ψ is basic and $\|\psi\|_{C_w^2(\bar{M})} \leq C$, for some constant C independent of ϵ.*

For any two function $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a unique C_w^2 solution $\varphi(t)$ of (1.7). Moreover, it is a limit of solutions of φ_ϵ of equation (1.12) such that $\Omega_{\varphi_\epsilon + 4\log r}$ is positive and bounded.

There are several geometric applications of Theorem 1. A direct consequence of it is that the infinite dimensional space $(\mathcal{H}, d)$ is a metric space. Theorem 1 guarantees the existence and uniqueness of C_w^2 geodesic for any two points in $\mathcal{H}$. One can define the length of C_w^2 geodesic as the geodesic distance d between two end points and can verify that the C_w^2 geodesic minimizes length over all possible curves between the two end points. As another geometric application, a $\mathcal{K}$ energy map $\mu : \mathcal{H} \to \mathbb{R}$ can be introduced as in the Kähler case. Theorem 1 implies μ is convex in $\mathcal{H}$. As in the Kähler case [5], this fact yields that the constant transversal scalar curvature metric (if it exists) realizes the global minimum of $\mathcal{K}$-energy if the first basic Chern class $C_1^B(M) \leq 0$. Furthermore, we will show that the constant transversal scalar curvature metric is unique in each basic Kähler class if $C_1^B(M) = 0$ or $C_1^B(M) < 0$. The details of these geometric applications of Theorem 1 can be found in Section 6. We also refer [9] for the role of geodesic equation in the discussion of the uniqueness of constant transversal scalar curvature metric on toric Sasakian manifolds.

The organization of the paper as follows. We derive the complex Monge-Ampére type equation on Kähler cone in the next section. Sections 3-5 devote to the a priori estimates of the equation, they are the core of this paper. The regularity of the geodesics will be used to prove $\mathcal{H}$ is a metric space in section 6, along with other geometric applications there. The proofs of the results in section 6 are given in the appendix.

2. A COMPLEX MONGE-AMPÉRE TYPE EQUATION ON KÄHLER CONE

We would like to transplant the geodesic equation (1.7) to a Dirichlet problem of complex Monge-Ampère type equation (1.10) on the Kähler cone. Let $C(M) = M \times \mathbb{R}^+$, $\bar{g} = dr^2 + r^2 g$, and (ξ, η, Φ, g) is a Sasakian structure on the manifold M. The almost complex structure on $C(M)$ defined by

(2.1) $$J(Y) = \Phi(Y) - \eta(Y)r\frac{\partial}{\partial r}, \quad J(r\frac{\partial}{\partial r}) = \xi,$$

makes $(C(M), \bar{g}, J)$ a Kähler manifolds since (ξ, η, Φ, g) is a Sasakian structure. In the following, pull back forms $p^*\eta$ and $p^*(d\eta)$ will be also denote by η and $d\eta$, where $p : C(M) \to M$ is the projective map. It's easy to check the following lemma, the proof can be found in [1] and [14].

Lemma 1. *The fundamental form $\bar{\omega}$ of the Kähler cone $(C(M), \bar{g})$ can be expressed by*

$$(2.2) \qquad \bar{\omega} = \frac{1}{2} r^2 d\eta + r dr \wedge \eta = \frac{1}{2} d(r^2 \eta) = \frac{1}{2} dd^c r^2.$$

As in the Kähler case, the Sasakian metric can locally be generated by a free real function of $2n$ variables [15]. This function is a Sasakian analogue of the Kähler potential for the Kähler geometry. More precisely, for any point q in M, there is a local **basic** function h and a local coordinate chart $(x, z^1, z^2, \cdots, z^n) \in \mathbb{R} \times C^n$ on a small neighborhood U around q, such that $\eta = dx - \sqrt{-1}(h_j dz^j - h_{\bar{j}} d\bar{z}^j)$ and $g = \eta \otimes \eta + 2h_{i\bar{j}} dz^i d\bar{z}^j$, where $h_i = \frac{\partial h}{\partial z^i}$, $h_{i\bar{j}} = \frac{\partial^2 h}{\partial z^i \partial \bar{z}^j}$. We can further assume that $h_i(q) = 0$, $h_{i\bar{j}}(q) = \delta^i_j$, and $d(h_{i\bar{j}})|_q = 0$. This can be achieved by a local change of coordinates through $(y, u^1, \cdots, u^n)$, where $y = x - \sqrt{-1}h_i(q)z^i + \sqrt{-1}h_{\bar{j}}(q)\bar{z}^j$ and $u^k = z^k$ for all $k = 1, \cdots, n$, and a change of potential function by $h^* = h - h_i(q)u^i - h_{\bar{j}}(q)\bar{u}^j$. This local coordinates also be called *normal coordinates* on Sasakian manifold.

For a normal local coordinate chart $(x, z^1, z^2, \cdots, z^n)$, set

$$(2.3) \qquad (z^1, z^2, \cdots z^n, w), \text{ on } U \times \mathbb{R}^+ \subset C(M), \text{ where } w = r + \sqrt{-1}x.$$

It should be pointed out that $(z^1, z^2, \cdots z^n, w)$ is not a holomorphic local coordinates of the complex manifold $C(M)$. Set

$$(2.4) \quad \begin{cases} X_j = \frac{\partial}{\partial z^j} + \sqrt{-1}h_j \frac{\partial}{\partial x}, \quad \bar{X}_j = \frac{\partial}{\partial \bar{z}^j} - \sqrt{-1}h_{\bar{j}} \frac{\partial}{\partial x}, \quad j = 1, \cdots, n; \\ X_{n+1} = \frac{1}{2}(\frac{\partial}{\partial r} - \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}), \quad \bar{X}_{n+1} = \frac{1}{2}(\frac{\partial}{\partial r} + \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}) \\ \theta^i = dz^i, \quad \theta^{n+1} = dr + \sqrt{-1}r\eta. \end{cases}$$

In this local coordinate chart, $\mathcal{D} \otimes C$ is spanned by X_i and $\bar{X}_i$ $i = 1, \cdots, n$, and

$$(2.5) \quad \begin{cases} \xi &= \frac{\partial}{\partial x}; \\ \eta &= dx - \sqrt{-1}(h_j dz^j - h_{\bar{j}} d\bar{z}^j); \\ \Phi &= \sqrt{-1}\{X_j \otimes dz^j - \bar{X}_j \otimes d\bar{z}^j\}; \\ g &= \eta \otimes \eta + 2h_{i\bar{j}} dz^i d\bar{z}^j, \end{cases}$$

$$(2.6) \quad \begin{array}{l} \Phi X_i = \sqrt{-1}X_i, \quad \Phi \bar{X}_i = -\sqrt{-1}\bar{X}_i, \\ [X_i, X_j] = [\bar{X}_i, \bar{X}_j] = [\xi, X_i] = [\xi, \bar{X}_i] = 0, \\ [X_i, \bar{X}_j] = -2\sqrt{-1}h_{i\bar{j}}\xi. \end{array}$$

$\{\eta, dz^i, d\bar{z}^j\}$ is the dual basis of $\{\frac{\partial}{\partial x}, X_i, \bar{X}_j\}$, and

$$(2.7) \qquad g^T = 2g^T_{i\bar{j}} dz^i d\bar{z}^j = 2h_{i\bar{j}} dz^i d\bar{z}^j, \quad d\eta = 2\sqrt{-1}h_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Proposition 1 is a special case $\epsilon = 0$ of the following.

Proposition 2. *The path φ_t connects $\varphi_0, \varphi_1 \in \mathcal{H}$ and satisfies equation (1.12) for some $\epsilon \geq 0$ if and only if ψ satisfies equation (1.11), where $f = r^2$, ψ and Ω_ψ defined as in (1.9), $\Omega_\psi|_{DC}$ is positive and $\psi|_{M \times \{1\}} = \varphi_0$, $\psi|_{M \times \{\frac{3}{2}\}} = \varphi_1 + 4\log\frac{3}{2}$.*

Proof. For any point p, we pick a local coordinate chart $(z_1, \cdots, z_n, w)$ as in (2.3) with properties (2.5)-(2.7). It is straightforward to check that

$$(2.8) \qquad \begin{array}{l} J(X_j) = \sqrt{-1}X_j, \quad J(\bar{X}_j) = -\sqrt{-1}\bar{X}_j, \\ J(X_{n+1}) = \sqrt{-1}X_{n+1}, \quad J(\bar{X}_{n+1}) = -\sqrt{-1}\bar{X}_{n+1}. \end{array}$$

$\{dz^j, d\bar{z}^j, dr + \sqrt{-1}r\eta, dr - \sqrt{-1}r\eta\}$ is the dual basis of $\{X_j, X_{\bar{j}}, X_{n+1}, \bar{X}_{n+1}\}$. Let $F(\cdot, r)$ be a smooth function on $M \times \mathbb{R}^+$, then we have,

$$\bar{\partial}F = (\bar{X}_i F)d\bar{z}^i + (\bar{X}_{n+1} F)(dr - \sqrt{-1}r\eta), \tag{2.9}$$

$$
\begin{aligned}
\partial\bar{\partial}F = {} & X_i \bar{X}_j F dz^i \wedge d\bar{z}^j + X_i \bar{X}_{n+1} F dz^i \wedge (dr - \sqrt{-1}r\eta) \\
& + X_{n+1} \bar{X}_j F (dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j \\
& + X_{n+1} \bar{X}_{n+1} F (dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta) \\
& - \sqrt{-1}r(\bar{X}_{n+1}F)d\eta + \tfrac{1}{2r}(\bar{X}_{n+1}F)(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta),
\end{aligned}
\tag{2.10}
$$

and

$$\partial\bar{\partial}r = -\sqrt{-1}\frac{1}{2}r d\eta + \frac{1}{4r}(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta). \tag{2.11}$$

From above, $\sqrt{-1}\partial\bar{\partial}r$ is a positive $(1,1)$-form on $M \times \mathbb{R}^+$. If $\frac{\partial}{\partial x}F = 0$, we have

$$
\begin{aligned}
\partial\bar{\partial}F - \frac{\partial F}{\partial r}\partial\bar{\partial}r = {} & \frac{\partial^2 F}{\partial z^i \partial \bar{z}^j}dz^i \wedge d\bar{z}^j + \frac{1}{2}\frac{\partial^2 F}{\partial z^i \partial r}dz^i \wedge (dr - \sqrt{-1}r\eta) \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial r \partial \bar{z}^j}(dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j + \frac{1}{4}\frac{\partial^2 F}{\partial r^2}(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta) \\
& - \sqrt{-1}\frac{1}{2}\frac{\partial F}{\partial r}r d\eta + \frac{1}{4r}\frac{\partial F}{\partial r}(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta) - \frac{\partial F}{\partial r}\partial\bar{\partial}r \\
= {} & \frac{\partial^2 F}{\partial z^i \partial \bar{z}^j}dz^i \wedge d\bar{z}^j + \frac{1}{2}\frac{\partial^2 F}{\partial z^i \partial r}dz^i \wedge (dr - \sqrt{-1}r\eta) \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial r \partial \bar{z}^j}(dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j + \frac{1}{4}\frac{\partial^2 F}{\partial r^2}(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta).
\end{aligned}
\tag{2.12}
$$

Let $\varphi_t : M \times [0,1] \to R$ be a path in the metric space H^0, define a function ψ on $\overline{M} = M \times [1, \frac{3}{2}] \subset C(M)$ defined as in (1.8). Since $\xi\psi \equiv 0$, for Ω_ψ defined as in (1.9),

$$
\begin{aligned}
\Omega_\psi = {} & \sqrt{-1}r^2\{(h_{i\bar{j}} + \tfrac{1}{2}\psi_{i\bar{j}})dz^i \wedge d\bar{z}^j + \tfrac{1}{4}\frac{\partial^2 \psi}{\partial z^i \partial r}dz^i \wedge (dr - \sqrt{-1}r\eta) \\
& + \tfrac{1}{4}\frac{\partial^2 \psi}{\partial r \partial \bar{z}^j}(dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j \\
& + (\tfrac{1}{8}\frac{\partial^2 \psi}{\partial r^2} + \tfrac{1}{2}r^{-2})(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta)\} \\
= {} & \sqrt{-1}r^2\{(h_{i\bar{j}} + \tfrac{1}{2}\varphi_{i\bar{j}})dz^i \wedge d\bar{z}^j + \tfrac{1}{2}\frac{\partial^2 \varphi}{\partial t \partial z^i}dz^i \wedge (dr - \sqrt{-1}r\eta) \\
& + \tfrac{1}{2}\frac{\partial^2 \varphi}{\partial t \partial \bar{z}^j}(dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j \\
& + \tfrac{1}{2}\frac{\partial^2 \varphi}{\partial t^2}(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta)\}
\end{aligned}
\tag{2.13}
$$

Hence

$$(\Omega_\psi)^{n+1} = 2^{-n}r^{2n+3}\left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial\varphi}{\partial t}|^2_{g_\varphi}\right)dr \wedge \eta \wedge (d\eta_\varphi)^n. \tag{2.14}$$

On the other hand, it's easy to check that

$$\omega^{n+1} = 2^{-n}r^{2n+1}dr \wedge \eta \wedge (d\eta)^n.$$

The proposition follows directly from (2.14). $\square$

We now want to choose appropriate subsolution for equation (1.11). Let $\psi_1, \psi_{\frac{3}{2}} \in \mathcal{H}$ be given boundary data on $\partial\overline{M}$, set $\psi_0 \in C^\infty(\overline{M})$ by

$$\psi_0(\cdot, r) = 2(\frac{3}{2} - r)\psi_1(\cdot) + 2(r - 1)\psi_{\frac{3}{2}}(\cdot) + m((2(r - 1) - \frac{1}{2})^2 - \frac{1}{4}), \tag{2.15}$$

where the positive constant m is chosen sufficiently large such that Ω_{ψ_0} is positive. Let

$$f_0 = \frac{(\Omega_{\psi_0})^{n+1}}{\bar{\omega}^{n+1}} > 0. \tag{2.16}$$

$\xi\psi_0 \equiv 0$ yields $\xi f_0 \equiv 0$.

We now fix given boundary data $\psi_1, \psi_{\frac{3}{2}} \in \mathcal{H}$. For any positive basic function f, set $f_s = sf + (1-s)f_0$ for each $0 \leq s \leq 1$. We consider the following Dirichlet problem

$$(2.17) \qquad \begin{cases} (\Omega_\psi)^{n+1} = f_s \bar{\omega}^{n+1}, & M \times (1, \frac{3}{2}), \\ \psi|_{r=1} = \psi_1, \\ \psi|_{r=\frac{3}{2}} = \psi_{\frac{3}{2}}. \end{cases}$$

In local coordinates, (2.17) can be written as

$$\det(\tilde{g}_{\alpha\bar{\beta}}) = f_s \det(g_{\alpha\bar{\beta}}), \quad \text{where } \tilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{r^2}{2}\psi_{\alpha\bar{\beta}} - \frac{r^2}{2}\frac{\partial\psi}{\partial r}r_{\alpha\bar{\beta}}.$$

Remark 1. *We note that for any $B \in \mathbb{R}$, $\Omega_{\psi+Br} = \Omega_\psi, \forall \psi \in \mathcal{H}$. Therefore, we may choose f_0 as large as we wish by picking m sufficient large (leaving the boundary data unchanged at the same time). For any given f, we may assume $f_0(Z) \geq f(Z), \forall Z \in \overline{M}$. ψ_0 is the unique solution to the equation (2.17) at $s = 1$. Also note that ψ_0 is a subsolution of (2.17) for each $0 \leq s \leq 1$.*

We will apply the method of continuity to solve (2.17). By Remark 1, we will assume (2.17) has a subsolution ψ_0. For the simplicity of notation, we will write f in place of f_s in (2.17). We will prove the following theorem.

Theorem 2. *For any smooth basic function $0 < f \in C_B^\infty(\overline{M})$ and basic boundary value ψ_0, there is a unique smooth solution ψ which is basic. Moreover, there exists constant C depending only on $\|f^{\frac{1}{n}}\|_{C^{1,1}(\overline{M})}$, $\|\psi_0\|_{C^{2,1}}$, and metric $\bar{g}$, such that*

$$(2.18) \qquad \|\psi\|_{C_w^2} \leq C.$$

Theorem 1 follows from Theorem 2.

We conclude this section with the following lemma.

Lemma 2. *Let ψ be a solution of the equation (2.17), and Ω_ψ is positive. If the boundary data of ψ is basic then $\xi\psi \equiv 0$ on $\overline{M}$. Moreover, the kernel of the linearized operator of equation (1.11) with null boundary data is trivial.*

Proof. Choose the same local coordinates $(z^1, \cdots\cdots, z^n, w)$ as in (2.3) with properties (2.5)-(2.7). $T^{1,0}\overline{M}$ is spanned by X_α, θ^α ($\alpha = 1, \cdots, n+1$) defined as in (2.4). Set

$$(2.19) \qquad \Omega_\psi = \sqrt{-1}\tilde{g}_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta,$$

where $i, j = 1, \cdots, n$, and $\alpha, \beta = 1, \cdots, n+1$. ψ is not assumed to be basic. We have

$$\begin{aligned} \Omega_\psi &= \tfrac{1}{2}\sqrt{-1}r^2\{(2h_{i\bar{j}} + X_i\bar{X}_j\psi + \sqrt{-1}\tfrac{\partial\psi}{\partial x})dz^i \wedge d\bar{z}^j \\ &\quad + \bar{X}_i\bar{X}_{n+1}\psi dz^i \wedge (dr - \sqrt{-1}r\eta) + X_{n+1}\bar{X}_j\psi(dr + \sqrt{-1}r\eta) \wedge d\bar{z}^j \\ &\quad + (X_{n+1}\bar{X}_{n+1}\psi + r^{-2} + \tfrac{1}{4r^2}\sqrt{-1}\tfrac{\partial\psi}{\partial x})(dr + \sqrt{-1}r\eta) \wedge (dr - \sqrt{-1}r\eta)\}, \end{aligned}$$

and,

$$\begin{aligned} &[X_i, \bar{X}_j] = -2\sqrt{-1}h_{i\bar{j}}\tfrac{\partial}{\partial x}, \quad [X_i, \bar{X}_{n+1}] = 0, \quad [X_{n+1}, \bar{X}_j] = 0, \\ &[X_{n+1}, \bar{X}_{n+1}] = -\tfrac{1}{2}\sqrt{-1}r^{-2}\tfrac{\partial}{\partial x}, \quad \tfrac{\partial}{\partial x}\tilde{g}_{\alpha\bar{\beta}} = \tfrac{r^2}{2}(X_\alpha\bar{X}_\beta\tfrac{\partial\psi}{\partial x} - \tfrac{1}{2}[X_\alpha, \bar{X}_\beta]\tfrac{\partial\psi}{\partial x}). \end{aligned}$$

Let $\tilde{\nabla}$ be the Chern connection of Hermitian metric $\tilde{g}(\cdot, \cdot) = \Omega_\psi(\cdot, J\cdot)$. Note that $\tilde{\nabla}\tilde{g} = 0$, $\tilde{\nabla}J = 0$, and the $(1, 1)$ part of the torsion vanishes,

$$\tilde{\nabla}_{X_\alpha}\bar{X}_\beta - \tilde{\nabla}_{\bar{X}_\beta}X_\alpha = [X_\alpha, \bar{X}_\beta], \quad \tilde{\nabla}_{X_\alpha}\bar{X}_\beta + \tilde{\nabla}_{\bar{X}_\beta}X_\alpha = \sqrt{-1}J([X_\alpha, \bar{X}_\beta])$$

Set $\omega^* = \frac{1}{2}r^2 p^* d\eta + \frac{1}{2}r dr \wedge \eta = \sqrt{-1}r\partial\bar{\partial}r$ and $g^*(\cdot,\cdot) = \omega^*(\cdot, J\cdot)$, we have

$$(2.20) \qquad \tilde{\nabla}_{X_\alpha}\bar{X}_\beta + \tilde{\nabla}_{\bar{X}_\beta}X_\alpha = -2r^{-1}g^*_{\alpha\bar{\beta}}\frac{\partial}{\partial r}.$$

It is straightforward to compute that

$$(2.21)\ 0 = \frac{\partial}{\partial x}(\log(f \det(g_{\alpha\bar{\beta}}))) = \frac{r^2}{4}\{\tilde{\triangle}(\xi\psi) + d(\xi\psi)(2r^{-1}\tilde{g}^{\alpha\bar{\beta}}g^*_{\alpha\bar{\beta}}\frac{\partial}{\partial r})\},$$

where $\tilde{\triangle}$ is the Laplacian of the Chern connection $\tilde{\nabla}$. Therefore, $\xi\psi$ satisfies homogeneous linear elliptic equation (2.21) with vanishing boundary data. It follows $\xi\psi \equiv 0$. The last assertion of the lemma follows from the same arguments. $\square$

3. C^1 ESTIMATE

This section and the next two sections will be devoted to the a priori estimates of solutions of equation (2.17). We start from C^0 estimate. We already have a subsolution to (1.11). We now construct a supersolution.

Let ρ be a smooth function on $\overline{M}$ such that

$$(3.1) \qquad \frac{r^2}{4}\triangle_{\bar{g}}\rho - \frac{r^2}{4}\triangle_{\bar{g}}r\frac{\partial\rho}{\partial r} + n + 1 = 0,$$

and satisfies the boundary condition $\rho(\cdot, 1) = \psi_1(\cdot)$, $\rho(\cdot, \frac{3}{2}) = \psi_{\frac{3}{2}}(\cdot)$. Therefore, ψ_0 and ρ are a subsolution and a supersolution of (2.17). The C^0 estimate is direct

$$(3.2) \qquad \psi_0 \leq \psi \leq \rho.$$

The next is the boundary gradient estimate.

Lemma 3. *Let ψ be a solution of the equation (2.17) and coincides with ψ_0 at the boundary $\partial\overline{M}$. Then there exists a constant C^* which depends only on ψ_0 and the metric $\bar{g}$ such that*

$$(3.3) \qquad |\frac{\partial\psi}{\partial r}(Z)| \leq C^*, \forall Z \in \overline{M}; \quad |d\psi|^2_{\bar{g}}(p) \leq C^*, \forall p \in \partial\overline{M}.$$

Proof. Since Ω_ψ is positive definite, if the boundary data of ψ is basic, it follows from (2.13) that $\frac{\partial^2\psi}{\partial r^2} > -4r^{-2}$ on $\overline{M}$. Together with (3.2), we obtain

$$(3.4) \qquad \frac{\partial\psi_0}{\partial r}(\cdot, 1) - \frac{4}{3} \leq \frac{\partial\psi}{\partial r}(\cdot, r) \leq \frac{\partial\psi_0}{\partial r}(\cdot, \frac{3}{2}) + \frac{4}{3}.$$

As $|d\psi|^2_{\bar{g}}(p) = |d\psi_0|^2_{\bar{g}} - (\frac{\partial\psi_0}{\partial r})^2 + (\frac{\partial\psi}{\partial r})^2$, $|d\psi|^2_{\bar{g}}(p)$ is under control. $\square$

The following is the global gradient estimate.

Proposition 3. *Suppose ψ is a solution of equation (2.17). Let $\phi = \psi - Br$, $B = \sup_{\overline{M}}\frac{\partial\psi}{\partial r}$, $W = |\partial\phi|^2_{\bar{g}}$, $L = \sup_{\overline{M}}|\phi|$. There exist positive constants A and C depending only on L, $\inf_{\overline{M}}R_{i\bar{i}j\bar{j}}$, $\|f^{\frac{1}{n+1}}\|_{C^1(\overline{M})}$, $\|r\|_{C^3}$, and $OSC_{\overline{M}}\frac{\partial\psi}{\partial r}$, if the maximum of $H = e^{Ae^{L-\phi}}W$ is achieved at an interior point p, then*

$$(3.5) \qquad H(p) \leq C.$$

Combing with Lemma 3, there exist a positive constant C_0 depending only on ρ, ψ_0, $\inf_{\overline{M}}R_{i\bar{i}j\bar{j}}$, $\|f^{\frac{1}{n+1}}\|_{C^1(\overline{M})}$ and $\|r\|_{C^3}$ such that

$$(3.6) \qquad |d\psi|^2_{\bar{g}}(Z) \leq C_0, \quad \forall Z \in \overline{M}.$$

Proof. As noted in Remark 1 that for $\phi = \psi - Br$, $\Omega_\phi = \Omega_\psi$ for any constant B. Since $\frac{\partial \psi}{\partial r}$ is bounded, set $B = \sup_{\overline{M}} \frac{\partial \psi}{\partial r}$ so that $\phi_r \leq 0$ and ϕ satisfies the same equation (2.17). We only need to prove (3.5). Pick a holomorphic normal coordinate system centered at p such that $\bar{g}_{\alpha\bar{\beta}}|_p = \delta_{\alpha\beta}$, $d\bar{g}_{\alpha\bar{\beta}}|_p = 0$, and $\tilde{g}_{\alpha\bar{\beta}}$ is diagonal at p, where $\tilde{g}_{\alpha\bar{\beta}} = \bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}\phi_{\alpha\bar{\beta}} - \frac{r^2}{2}\frac{\partial\phi}{\partial r}r_{\alpha\bar{\beta}}$. We may assume that $W(p) \geq 1$.

Differentiate $\log H$ at p,

$$(3.7) \qquad \frac{W_\alpha}{W} - Ae^{L-\phi}\phi_\alpha = 0, \quad \frac{W_{\bar\alpha}}{W} - Ae^{L-\phi}\phi_{\bar\alpha} = 0.$$

We also have

$$W_\alpha = \phi_{\beta\alpha}\phi_{\bar\beta} + \phi_{\bar\beta\alpha}\phi_\beta, \quad W_{\bar\alpha} = \phi_{\beta\bar\alpha}\phi_{\bar\beta} + \phi_{\bar\beta\bar\alpha}\phi_\beta$$

$$W_{\alpha\bar\alpha} = \bar{g}^{\beta\bar\delta}_{,\alpha\bar\alpha}\phi_\beta\phi_{\bar\delta} + \sum(|\phi_{\beta\alpha}|^2 + |\phi_{\beta\bar\alpha}|^2) + \phi_\beta\phi_{\bar\beta\alpha\bar\alpha} + \phi_{\bar\beta}\phi_{\beta\alpha\bar\alpha},$$

$$\begin{aligned}
|W_\alpha|^2 &= (\phi_{\bar\beta}\phi_{\beta\alpha})(\phi_\eta\phi_{\bar\eta\bar\alpha}) + (\phi_\beta\phi_{\bar\beta\alpha})(\phi_{\bar\eta}\phi_{\eta\bar\alpha}) \\
&\quad + (\phi_{\bar\beta}\phi_{\beta\alpha})(\phi_{\bar\eta}\phi_{\eta\bar\alpha}) + (\phi_\beta\phi_{\bar\beta\alpha})(\phi_\eta\phi_{\bar\eta\bar\alpha}) \\
&= |\phi_{\bar\beta}\phi_{\beta\alpha}|^2 - (\phi_\beta\phi_{\bar\beta\alpha})(\phi_{\bar\eta}\phi_{\eta\bar\alpha}) \\
&\quad + Ae^{L-\phi}W(\phi_\beta\phi_{\bar\beta\alpha}\phi_{\bar\alpha} + \phi_{\bar\eta}\phi_{\eta\bar\alpha}\phi_\alpha).
\end{aligned}$$

Pick $A \geq 1$ sufficient large, so that $(\bar{g}^{\eta\bar\delta}_{\alpha\bar\beta}\phi_\eta\phi_{\bar\delta}W^{-1} + \frac{2}{9}A\bar{g}_{\alpha\bar\beta}) \geq 0$. Since $(\tilde{g}_{\alpha\bar\beta}) > 0$, by the assumption, at p,

$$\begin{aligned}
(3.8) \qquad 0 &\geq F^{\alpha\bar\beta}(\log H)_{\alpha\bar\beta} = F^{\alpha\bar\alpha}(\log H)_{\alpha\bar\alpha} \\
&= F^{\alpha\bar\alpha}\{W_{\alpha\bar\alpha}W^{-1} - |W_\alpha|^2W^{-2} - Ae^{L-\phi}(\phi_{\alpha\bar\alpha} - |\phi_\alpha|^2)\} \\
&= F^{\alpha\bar\alpha}\{\bar{g}^{\eta\bar\delta}_{\alpha\bar\alpha}\phi_\eta\phi_{\bar\delta}W^{-1} - Ae^{L-\phi}(\phi_{\alpha\bar\alpha} - |\phi_\alpha|^2) \\
&\quad - Ae^{L-\phi}W^{-1}(\phi_\beta\phi_{\bar\beta\alpha}\phi_{\bar\alpha} + \phi_{\bar\eta}\phi_{\eta\bar\alpha}\phi_\alpha) \\
&\quad + [W^{-1}(\sum_\beta |\phi_{\beta\alpha}|^2) - W^{-2}|\sum_\beta \phi_{\bar\beta}\phi_{\beta\alpha}|^2] \\
&\quad + W^{-1}|\phi_{\beta\bar\alpha}|^2 + W^{-2}|\sum_\eta \phi_{\bar\eta}\phi_{\eta\bar\alpha}|^2 \\
&\quad + \sum_\beta W^{-1}(\phi_\beta\phi_{\bar\beta\alpha\bar\alpha} + \phi_{\bar\beta}\phi_{\beta\alpha\bar\alpha})\},
\end{aligned}$$

where $F^{\alpha\bar\beta}$ is the (α, β)th cofactor of the matrix $(\tilde{g}_{\alpha\bar\beta})$. On the other hand,

$$(3.9) \qquad \phi_{\alpha\bar\beta} = \psi_{\alpha\bar\beta} - Br_{\alpha\bar\beta} = 2r^{-2}(\tilde{g}_{\alpha\bar\beta} - g_{\alpha\bar\beta}) + \frac{\partial\phi}{\partial r}r_{\alpha\bar\beta},$$

$$\begin{aligned}
(3.10) \qquad \phi_\beta\phi_{\bar\beta\alpha}\phi_{\bar\alpha} &= \phi_\beta\phi_{\bar\alpha}(2r^{-2}(\tilde{g}_{\alpha\bar\beta} - g_{\alpha\bar\beta}) + \frac{\partial\phi}{\partial r}r_{\alpha\bar\beta}), \\
\phi_{\bar\eta}\phi_{\eta\bar\alpha}\phi_\alpha &= \phi_{\bar\eta}\phi_\alpha(2r^{-2}(\tilde{g}_{\eta\bar\alpha} - g_{\eta\bar\alpha}) + \frac{\partial\phi}{\partial r}r_{\eta\bar\alpha}),
\end{aligned}$$

$$\begin{aligned}
(3.11) \qquad &\sum_\beta W^{-1}(\phi_\beta\phi_{\bar\beta\alpha\bar\alpha} + \phi_{\bar\beta}\phi_{\beta\alpha\bar\alpha}) \\
&= -4r^{-3}W^{-1}(\phi_\beta r_{\bar\beta} + \phi_{\bar\beta}r_\beta)(\tilde{g}_{\alpha\bar\alpha} - g_{\alpha\bar\alpha}) \\
&\quad + 2r^{-2}W^{-1}(\phi_\beta\tilde{g}_{\alpha\bar\alpha,\bar\beta} + \phi_{\bar\beta}\tilde{g}_{\alpha\bar\alpha,\beta}) \\
&\quad + W^{-1}\phi_r(\phi_\beta r_{\alpha\bar\alpha\bar\beta} + \phi_{\bar\beta}r_{\alpha\bar\alpha\beta}) \\
&\quad + W^{-1}r_{\alpha\bar\alpha}((\frac{\partial\phi}{\partial r})_{\bar\beta}\phi_\beta + (\frac{\partial\phi}{\partial r})_\beta\phi_{\bar\beta})
\end{aligned}$$

and

$$
\begin{aligned}
& W^{-1}r_{\alpha\bar\alpha}\big((\tfrac{\partial\phi}{\partial r})_{\bar\beta}\phi_\beta + (\tfrac{\partial\phi}{\partial r})_\beta\phi_{\bar\beta}\big) = W^{-1}r_{\alpha\bar\alpha}\big((\tfrac{\partial\phi}{\partial r})_{\bar\beta}\phi_\beta + (\tfrac{\partial\phi}{\partial r})_\beta\phi_{\bar\beta}\big) \\
=\ & W^{-1}r_{\alpha\bar\alpha}\big(\tfrac{\partial\phi_{\bar\beta}}{\partial r}\phi_\beta + \tfrac{\partial\phi_\beta}{\partial r}\phi_{\bar\beta}\big) \\
& + W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_{\bar\beta}\phi_{\bar\delta}\phi_\beta + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_{\bar\beta}\phi_\eta\phi_\beta\big) \\
& + W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_\beta\phi_{\bar\delta}\phi_{\bar\beta} + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_\beta\phi_\eta\phi_{\bar\beta}\big) \\
=\ & Ae^{L-\phi}\phi_r r_{\alpha\bar\alpha} \\
& + W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_{\bar\beta}\phi_{\bar\delta}\phi_\beta + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_{\bar\beta}\phi_\eta\phi_\beta\big) \\
& + W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_\beta\phi_{\bar\delta}\phi_{\bar\beta} + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_\beta\phi_\eta\phi_{\bar\beta}\big)
\end{aligned}
\tag{3.12}
$$

Hence,

$$
\begin{aligned}
& -Ae^{L-\phi}\phi_{\alpha\bar\alpha} + \sum_\beta W^{-1}(\phi_\beta\phi_{\bar\beta\alpha\bar\alpha} + \phi_{\bar\beta}\phi_{\beta\alpha\bar\alpha}) \\
=\ & Ae^{L-\phi}2r^{-2}(g_{\alpha\bar\alpha} - \tilde g_{\alpha\bar\alpha}) \\
& -4r^{-3}W^{-1}(\phi_\beta r_{\bar\beta} + \phi_{\bar\beta}r_\beta)(\tilde g_{\alpha\bar\alpha} - g_{\alpha\bar\alpha}) \\
& +2r^{-2}W^{-1}(\phi_\beta\tilde g_{\alpha\bar\alpha,\bar\beta} + \phi_{\bar\beta}\tilde g_{\alpha\bar\alpha,\beta}) \\
& +W^{-1}\phi_r(\phi_\beta r_{\alpha\bar\alpha\bar\beta} + \phi_{\bar\beta}r_{\alpha\bar\alpha\beta}) \\
& +W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_{\bar\beta}\phi_{\bar\delta}\phi_\beta + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_{\bar\beta}\phi_\eta\phi_\beta\big) \\
& +W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_\beta\phi_{\bar\delta}\phi_{\bar\beta} + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_\beta\phi_\eta\phi_{\bar\beta}\big).
\end{aligned}
$$

As $F^{\alpha\bar\alpha}\tilde g_{\alpha\bar\alpha,\beta} = f_\beta$ and $\phi_r \le 0$, at point p,

$$
\begin{aligned}
0 \ge\ & F^{\alpha\bar\beta}(\log H)_{\alpha\bar\beta} \\
=\ & F^{\alpha\bar\alpha}\{\bar g^{\eta\bar\delta}_{\alpha\bar\alpha}\phi_\eta\phi_{\bar\delta}W^{-1} + Ae^{L-\phi}(2r^{-2}\bar g_{\alpha\bar\alpha} + |\phi_\alpha|^2) \\
& -2Ae^{L-\phi}r^{-2}\tilde g_{\alpha\bar\alpha}(1 + 2w^{-1}|\phi_\alpha|^2) + 4Ae^{L-\phi}r^{-2}\bar g_{\alpha\bar\alpha}W^{-1}|\phi_\alpha|^2 \\
& -Ae^{L-\phi}W^{-1}\phi_r(\phi_\beta r_{\bar\beta\alpha}\phi_{\bar\alpha} + \phi_{\bar\eta}r_{\eta\bar\alpha}\phi_\alpha) \\
& +[W^{-1}(\textstyle\sum_\beta|\phi_{\beta\alpha}|^2) - W^{-2}|\sum_\beta\phi_{\bar\beta}\phi_{\beta\alpha}|^2] \\
& +W^{-1}|\phi_{\beta\bar\alpha}|^2 + W^{-2}|\sum_\eta\phi_{\bar\eta}\phi_{\eta\bar\alpha}|^2 \\
& +2r^{-2}W^{-1}(\phi_\beta\tilde g_{\alpha\bar\alpha,\bar\beta} + \phi_{\bar\beta}\tilde g_{\alpha\bar\alpha,\beta}) \\
& -4r^{-3}W^{-1}(\phi_\beta r_{\bar\beta} + \phi_{\bar\beta}r_\beta)(\tilde g_{\alpha\bar\alpha} - \bar g_{\alpha\bar\alpha}) \\
& +W^{-1}\phi_r(\phi_\beta r_{\alpha\bar\alpha\bar\beta} + \phi_{\bar\beta}r_{\alpha\bar\alpha\beta}) \\
& +W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_{\bar\beta}\phi_{\bar\delta}\phi_\beta + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_{\bar\beta}\phi_\eta\phi_\beta\big) \\
& +W^{-1}r_{\alpha\bar\alpha}\big((\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial z^\eta})_\beta\phi_{\bar\delta}\phi_{\bar\beta} + (\bar g^{\eta\bar\delta}\tfrac{\partial r}{\partial \bar z^\delta})_\beta\phi_\eta\phi_{\bar\beta}\big)\} \\
\ge\ & F^{\alpha\bar\alpha}\{(\bar g^{\eta\bar\delta}_{\alpha\bar\alpha}\phi_\eta\phi_{\bar\delta}W^{-1} + Ae^{L-\phi}\tfrac12 r^{-2}\bar g_{\alpha\bar\alpha}) + Ae^{L-\phi}(r^{-2}\bar g_{\alpha\bar\alpha} + |\phi_\alpha|^2) \\
& -6Ae^{L-\phi}r^{-2}\tilde g_{\alpha\bar\alpha} + Ae^{L-\phi}\tfrac12 r^{-2}\bar g_{\alpha\bar\alpha} - 4r^{-3}W^{-\frac12}\bar g_{\alpha\bar\alpha} - 2\|r\|_{C^2}r_{\alpha\bar\alpha} \\
& 2W^{-\frac12}\phi_r\|r\|_{C^3} - 4r^{-3}W^{-\frac12}\tilde g_{\alpha\bar\alpha}\} - 2r^{-2}W^{-\frac12}|\nabla f|_{\bar g},
\end{aligned}
$$

By choosing A sufficiently large, such that

$$
A\frac12 r^{-2} - 4r^{-3} + 2\phi_r\|r\|_{C^3} - 2\|r\|_{C^2}\triangle_{\bar g}r \ge 0,
\tag{3.13}
$$

where A depending only on r, $OSC(\psi_r)$, $\|r\|_{C^3}$ and the lower bound of the holomorphic bisectional curvature of $(\overline M, g)$. Since $\{\tilde g_{\alpha\bar\beta}\}$ is diagonal at point p, we may arrange $\tilde g_{1\bar1} \le \cdots \le \tilde g_{n+1\overline{n+1}}$. Thus,

$$
\begin{aligned}
& \textstyle\sum_\alpha F^{\alpha\bar\alpha}(\bar g_{\alpha\bar\alpha} + |\phi_\alpha|^2) \\
=\ & \det(\tilde g_{\gamma\bar\beta})\textstyle\sum_\alpha \tilde g^{\alpha\bar\alpha}(\bar g_{\alpha\bar\alpha} + |\phi_\alpha|^2) \\
\ge\ & \det(\tilde g_{\gamma\bar\beta})\{\textstyle\sum_\alpha \tilde g^{\alpha\bar\alpha} + \tilde g^{n+1\overline{n+1}}W\} \\
\ge\ & \det(\tilde g_{\gamma\bar\beta})\{\textstyle\sum_{i=1}^n \tilde g^{i\bar i} + \tilde g^{n+1\overline{n+1}}(1+W)\} \\
\ge\ & (n+1)(\det(\tilde g_{\gamma\bar\beta}))^{1-\frac{1}{n+1}}(1+W)^{\frac{1}{n+1}}.
\end{aligned}
\tag{3.14}
$$

Note that $F^{\alpha\bar{\alpha}}\tilde{g}_{\alpha\bar{\alpha}} = (n+1)f$, the above inequality yields

$$
\begin{aligned}
0 &\geq Ae^{L-\phi}r^{-2}\{\textstyle\sum_\alpha F^{\alpha\bar{\alpha}}(\bar{g}_{\alpha\bar{\alpha}} + |\phi_\alpha|^2) - 10(n+1)f\} - 2r^{-2}W^{-\frac{1}{2}}|\nabla f| \\
&\geq Ae^{L-\phi}r^{-2}((f)^{1-\frac{1}{n+1}}W^{\frac{1}{n+1}} - 10(n+1)f) - 2r^{-2}W^{-\frac{1}{2}}|\nabla f| \\
(3.15) \quad &= r^{-2}(f)^{1-\frac{1}{n+1}}\{Ae^{L-\phi}(W^{\frac{1}{n+1}} - 10(n+1)(f)^{\frac{1}{n+1}}) \\
&\quad -2(n+1)W^{-\frac{1}{2}}|\nabla(f)^{\frac{1}{n+1}}|\} \\
&= r^{-2}(f)^{1-\frac{1}{n+1}}\{Ae^{L-\phi}(\tfrac{1}{2}W^{\frac{1}{n+1}} - 10(n+1)f^{\frac{1}{n+1}}) \\
&\quad +(Ae^{L-\phi}\tfrac{1}{2}W^{\frac{1}{n+1}} - 2(n+1)W^{-\frac{1}{2}}|\nabla f^{\frac{1}{n+1}}|)\}.
\end{aligned}
$$

Now (3.5) follows directly. $\square$

4. $C^{1,1}$ Boundary estimate

$C^{1,1}$ boundary estimates will be proved in this section. The construction of barriers follows from B. Guan [16] (in the real case, this method was introduced by Hoffman-Rosenberg-Spruck [20] and Guan-Spruck in [17]). Let ψ be a solution of the equation (2.17), we want to obtain second derivative estimates of ψ on the boundary $\partial\overline{M} = M \times \{1\} \cup M \times \{\frac{3}{2}\}$. For any point $p = (q,1) \in M \times \{1\}$ (or $p = (q,\frac{3}{2}) \in M \times \{\frac{3}{2}\}$), we may pick a local coordinate chart as in (2.3) with properties (2.5)-(2.7). Furthermore, we may assume

$$
(4.1) \qquad \frac{1}{4}\delta_{ij} \leq h_{i\bar{j}}(z) \leq \delta_{ij}, \quad \sum_{i=1}^n |h_i|^2(z) \leq 1, \quad \forall z \in U,
$$

where h is a local real basic function, $\eta = dx - \sqrt{-1}(h_i dz^i - h_{\bar{j}}d\bar{z}^j)$, $z^i = x^i + \sqrt{-1}y^i$ and $i,j = 1,\cdots,n$. Now, by setting $V = U \times [1, 1+\delta]$ (or $V = U \times [\frac{3}{2} - \delta, \frac{3}{2}]$), we have a local coordinates $(r,x,z^1,\cdots,z^n)$ on V such that $\partial\overline{M} \cap V = \{r = 1\}$ (or $r = \frac{3}{2}$). For X_α, θ^β defined in (2.4), $\{X_1,\cdots,X_j,\cdots X_n, X_{n+1}\}$ is a basis of $T^{1,0}(\overline{M})$, and $\{\theta^1,\cdots,\theta^{n+1}\}$ is the dual basis.

The Kähler form $\bar{\omega}$ of $(\overline{M}, \bar{g})$ can be written as

$$
(4.2) \qquad \bar{\omega} = \sqrt{-1}(r^2 h_{i\bar{j}}\theta^i \wedge \bar{\theta}^j + \frac{1}{2}\theta^{n+1} \wedge \bar{\theta}^{n+1}) = \sqrt{-1}\bar{g}_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta,
$$

and

$$
\begin{aligned}
(4.3) \quad \Omega_\psi &= \sqrt{-1}\{(r^2 h_{i\bar{j}} + \frac{r^2}{2}\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j})\theta^i \wedge \bar{\theta}^j + \frac{r^2}{4}\frac{\partial^2\psi}{\partial z^i\partial r}\theta^i \wedge \bar{\theta}^{n+1} \\
&\quad + \frac{r^2}{4}\frac{\partial^2\psi}{\partial r\partial\bar{z}^j}\theta^{n+1} \wedge \bar{\theta}^j + (\frac{r^2}{8}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{2})\theta^{n+1} \wedge \bar{\theta}^{n+1}\} \\
&= \sqrt{-1}(\bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}X_\alpha\bar{X}_\beta\psi)\theta^\alpha \wedge \bar{\theta}^\beta.
\end{aligned}
$$

Since $2(\bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}X_\alpha\bar{X}_\beta\psi_0)\theta^\alpha\bar{\theta}^\beta$ is a Hermitian metric on $\overline{M}$, there exists a constant $0 < a_0 < 1$ such that

$$
(4.4) \qquad a_0\bar{g}_{\alpha\bar{\beta}} < \bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}X_\alpha\bar{X}_\beta\psi_0 < \frac{1}{a_0}\bar{g}_{\alpha\bar{\beta}}
$$

in $\overline{M}$. In the neighborhood V of p, we have

$$
(4.5) \qquad \frac{1}{4}a_0\delta_{\alpha\beta} < \bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}X_\alpha\bar{X}_\beta\psi_0 < \frac{9}{4}\frac{1}{a_0}\delta_{\alpha\beta}.
$$

Let $\triangle_\psi$ be the canonical Laplacian corresponding with the Chern connection determined by the Hermitian metric $\Omega_\psi = \sqrt{-1}\tilde{g}_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta$ on $\overline{M}$. In Kähler case, this canonical Lapalacian is same as the standard Levi-Civita Laplacian. In general

Hermitian case they are different, the difference of two Laplacian is a first order linear differential operator. In the above local coordinates,

$$(4.6) \quad \begin{aligned} \tfrac{1}{2}\triangle_\psi u &= \ <\sqrt{-1}\partial\bar\partial u, \Omega_\psi >_\psi \\ &= \ -<(X_\gamma \bar X_\delta u)\theta^\gamma \wedge \bar\theta^\delta + 2(\bar X_{n+1}u)\partial\bar\partial r, \tilde g_{\alpha\bar\beta}\theta^\alpha \wedge \bar\theta^\beta >_\psi \\ &= \ (\tilde g)^{\alpha\bar\beta}(X_\alpha \bar X_\beta u) + (\bar X_{n+1}u)\triangle_\psi r, \end{aligned}$$

where $(\tilde g)^{\alpha\bar\beta}\tilde g_{\gamma\bar\beta} = \delta_{\alpha\gamma}$. Define differential operator $\mathbf{L}$ as

$$(4.7) \qquad\qquad \mathbf{L}u = \frac{1}{2}\triangle_\psi u - \frac{1}{2}\frac{\partial u}{\partial r}\triangle_\psi r$$

for all $u \in C^\infty(\overline M)$.

We now assume $f^{\frac{1}{n}} \in C^{1,1}$. This implies $|\nabla f^{\frac{1}{n}}(Z)| \leq Cf^{\frac{1}{2n}}(Z), \forall Z \in \overline M$. Since $\sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha} \geq (n+1)f^{-\frac{1}{n+1}}$, we have

$$(4.8) \qquad\qquad \frac{|\nabla f^{\frac{1}{n}}|}{f}(Z) \leq Cf^{-\frac{1}{2n}}(Z) \leq C\sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha}(Z), \forall Z \in \overline M.$$

Let D be any locally defined constant linear first order operator (with respect to the coordinate chart we chosen) near the boundary (e.g., $D = \pm\frac{\partial}{\partial x^i}, \pm\frac{\partial}{\partial y^i}$ for any $1 \leq i \leq n$). Differentiating both side of equation (2.17) by D, by (4.8),

$$(4.9) \quad \begin{aligned} \mathbf{L}D(\psi - \psi_0) &= \ \tfrac{1}{2}\triangle_\psi D(\psi - \psi_0) - \tfrac{1}{2}\frac{\partial D(\psi-\psi_0)}{\partial r}\triangle_\psi r \\ &= \ (\tilde g)^{\alpha\bar\beta}X_\alpha \bar X_\beta D(\psi - \psi_0) \\ &= \ 2r^{-2}(\tilde g)^{\alpha\bar\beta}D\{\tfrac{r^2}{2}X_\alpha \bar X_\beta(\psi - \psi_0)\} \\ &= \ 2r^{-2}(\tilde g)^{\alpha\bar\beta}D\{\tilde g_{\alpha\bar\beta} - (\bar g_{\alpha\bar\beta} + \tfrac{r^2}{2}X_\alpha \bar X_\beta(\psi_0))\} \\ &= \ 2r^{-2}D(\log f + \log\det(\bar g_{\alpha\bar\beta})) \\ &\quad\ -2r^{-2}(\tilde g)^{\alpha\bar\beta}D(\bar g_{\alpha\bar\beta} + \tfrac{r^2}{2}X_\alpha \bar X_\beta(\psi_0)) \\ &\leq \ \tilde C_1(1 + \tfrac{|\nabla f|}{f} + \sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha}) \\ &\leq \ C_1(1 + \sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha}), \end{aligned}$$

where constant C_1 depend only on ψ_0, $\|f^{\frac{1}{n}}\|_{C^{1,1}}$ and the metric $\bar g$. (Here we have used the properties that ψ and ψ_0 are basic, $[D, X_{n+1}] = 0$)

Now, choose a barrier function of the form

$$(4.10) \qquad\qquad v = (\psi - \psi_0) + b(\rho - \psi_0) - N(r - 1)^2$$

if $p \in M \times \{1\}$ (or $v = (\psi - \psi_0) + b(\rho - \psi_0) - N(r - \frac{3}{2})^2$ if $p \in M \times \{\frac{3}{2}\}$).

Lemma 4. *For N sufficiently large and b, δ_0 sufficiently small, we have*

$$(4.11) \qquad\qquad \mathbf{L}v \leq -\frac{a_0}{9}(1 + \sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha})$$

in $U \times [1, \frac{3}{2}]$, and $v \geq 0$ in $M \times [1, 1 + \delta_0]$ (or in $M \times [\frac{3}{2} - \delta_0, \frac{3}{2}]$), where constants only depend on ψ_0, ρ, $\|f^{\frac{1}{n}}\|_{C^{1,1}}$, and $\bar g$.

Proof. By assumption,

$$(4.12) \quad \begin{aligned} \mathbf{L}(\psi - \psi_0) &= \ (\tilde g)^{\alpha\bar\beta}X_\alpha \bar X_\beta(\psi - \psi_0) \\ &= \ 2r^{-2}(\tilde g)^{\alpha\bar\beta}\{\tilde g_{\alpha\bar\beta} - (\bar g_{\alpha\bar\beta} + \tfrac{r^2}{2}X_\alpha \bar X_\beta(\psi_0))\} \\ &\leq \ 2r^{-2}(n + 1 - \tfrac{a_0}{4}\sum_{\alpha=1}^{n+1}(\tilde g)^{\alpha\bar\alpha}), \end{aligned}$$

and

$$
(4.13) \qquad \mathbf{L}(\rho - \psi_0) \le C_2 (1 + \sum_{\alpha=1}^{n+1} (\tilde{g})^{\alpha \bar{\alpha}})
$$

where constant C_2 only depend on ρ and the metric $\bar{g}$. Then,

$$
(4.14) \qquad
\begin{aligned}
\mathbf{L}v &= \mathbf{L}(\psi - \psi_0) + b\mathbf{L}(\rho - \psi_0) - \tfrac{1}{2}N(\tilde{g})^{n+1\overline{n+1}} \\
&\le 2r^{-2}(n+1 - \tfrac{a_0}{4}\sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}) + bC_2(1 + \sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}) \\
&\quad - \tfrac{1}{2}N(\tilde{g})^{n+1\overline{n+1}}.
\end{aligned}
$$

Suppose $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n+1}$ are eigenvalues of $(\tilde{g}_{\alpha\bar{\alpha}})$. It follows that

$$
(4.15) \qquad \sum_{\alpha=1}^{n+1} (\tilde{g})^{\alpha\bar{\alpha}} = \sum_{\alpha=1}^{n+1} \lambda_\alpha^{-1}, \quad (\tilde{g})^{n+1\overline{n+1}} \ge \lambda_{n+1}^{-1}.
$$

Thus

$$
(4.16) \qquad
\begin{aligned}
& r^{-2}\tfrac{a_0}{8}\sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}} + \tfrac{1}{2}N(\tilde{g})^{n+1\overline{n+1}} \\
\ge \ & r^{-2}\tfrac{a_0}{4}\sum_{\alpha=1}^{n+1}\lambda_\alpha^{-1} + \tfrac{1}{2}N\lambda_{n+1}^{-1} \\
\ge \ & (n+1)(r^{-2}\tfrac{a_0}{4})^{\frac{n}{n+1}} N^{\frac{1}{n+1}}(\lambda_1 \cdots \lambda_{n+1})^{\frac{1}{n+1}} \\
\ge \ & C_3 N^{\frac{1}{n+1}},
\end{aligned}
$$

where positive constant C_3 depends only on f and $(\overline{M}, \bar{g})$. Choose N large enough so that

$$
(4.17) \qquad -C_3 N^{\frac{1}{n+1}} + 2r^{-2}(n+1) + bC_2 \le \frac{a_0}{9},
$$

and choose b small enough so that $bC_2 \le \frac{a_0}{18}$. Then, on $U \times [1, \tfrac{3}{2}]$, we have

$$
\mathbf{L}v \le -\frac{a_0}{9}(1 + \sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}).
$$

By the definition of function ρ,

$$
(4.18) \qquad
\begin{aligned}
& \triangle_{\bar{g}}(\rho - \psi_0) - \triangle_{\bar{g}}r \cdot \tfrac{\partial}{\partial r}(\rho - \psi_0) \\
= \ & (\triangle_{\bar{g}}\rho - \triangle_{\bar{g}}r \cdot \tfrac{\partial}{\partial r}\rho) - (\triangle_{\bar{g}}\psi_0 - \triangle_{\bar{g}}r \cdot \tfrac{\partial}{\partial r}\psi_0) \\
= \ & -4r^{-2}\{n+1 + \tfrac{r^2}{2}\bar{g}^{\alpha\bar{\beta}}X_\alpha \bar{X}_\beta \psi_0\} \\
= \ & -4r^{-2}\bar{g}^{\alpha\bar{\beta}}(\bar{g}_{\alpha\bar{\beta}} + \tfrac{r^2}{2}X_\alpha \bar{X}_\beta \psi_0) \\
\le \ & -4r^{-2}(n+1)a_0.
\end{aligned}
$$

On the boundary $\partial \overline{M}$, since ρ coincide with ψ_0 on the boundary,

$$
(4.19) \qquad
\begin{aligned}
\tfrac{\partial^2}{\partial r^2}(\rho - \psi_0) &= 2\bar{g}^{\alpha\bar{\beta}}X_\alpha \bar{X}_\beta (\rho - \psi_0) \\
&= \triangle_{\bar{g}}(\rho - \psi_0) - \triangle_{\bar{g}}r \cdot \tfrac{\partial}{\partial r}(\rho - \psi_0) \\
&\le -4r^{-2}(n+1)a_0 < -a_0.
\end{aligned}
$$

As $\psi_0 \le \rho$ on $\overline{M}$, it's easy to show that $\frac{\partial(\rho - \psi_0)}{\partial r}(q, 1) > 0$ and $\frac{\partial(\rho - \psi_0)}{\partial r}(q, \tfrac{3}{2}) > 0$ for every $q \in M$. Therefore, there exists a positive constant C_4 depending only on ρ, ψ_0 and $\bar{g}$ such that $\rho - \psi_0 > C_4(r-1)$ near $M \times \{1\}$ and $\rho - \psi_0 > C_4(\tfrac{3}{2} - r)$ near $M \times \{\tfrac{3}{2}\}$. From now on we fix N, and choose δ_0 small enough so that

$$
(4.20) \qquad b(\rho - \psi_0) - N(r-1)^2 \ge (bC_4 - N\delta)(r-1) \ge 0,
$$

in $M \times [1, 1 + \delta_0]$, and

$$
(4.21) \qquad b(\rho - \psi_0) - N(\tfrac{3}{2} - r)^2 \ge (bC_4 - N\delta)(\tfrac{3}{2} - r) \ge 0,
$$

in $M \times [\frac{3}{2} - \delta_0, \frac{3}{2}]$. Then $v \geq 0$ in $M \times [1, 1 + \delta_0]$ (or in $M \times [\frac{3}{2} - \delta_0, \frac{3}{2}]$). $\square$

Lemma 5. *There exists a constant C_5 which depends only on $(\overline{M}, \bar{g})$, ψ_0, $\|f^{\frac{1}{n}}\|_{C^{1,1}}$, and ρ such that*

$$(4.22) \qquad |\frac{\partial^2 \psi}{\partial z^i \partial r}(p)| \leq C_5 \max_{\overline{M}}(|d\psi|_{\bar{g}} + 1)$$

for every $p \in \partial \overline{M}$.

Proof. Suppose $p = (q, 1)$ (or $p = (q, \frac{3}{2})$), we can choose δ small enough such that $B_{2\delta}(0) = \{(x, z^1, \cdots, z^n) : x^2 + \sum |z^i|^2 \leq 4\delta\} \subset U$ and $2\delta \leq \delta_0$. The constant δ depends only on $(\overline{M}, \bar{g})$ ψ_0 and ρ. Let $V_\delta = \{(r, x, z^1, \cdots, z^n) : (r - 1)^2 + x^2 + \sum |z^i|^2 \leq \delta\} \cap \overline{M}$ if $p = (q, 1)$ (or $V_\delta = \{(r, x, z^1, \cdots, z^n) : (\frac{3}{2} - r)^2 + x^2 + \sum |z^i|^2 \leq \delta\} \cap \overline{M}$ if $p = (q, \frac{3}{2})$). Let $A = \max_{\overline{M}}(|d\psi|_{\bar{g}} + 1)$. Choose d_1, d_2 as big multiples of A such that $d_2 \delta^2 - |D(\psi - \psi_0)| > 0$. Consider $\mu = d_1 v + d_2(x^2 + (r - 1)^2 + \sum |z^i|^2) + D(\psi - \psi_0)$ (or $\mu = d_1 v + d_2(x^2 + (\frac{3}{2} - r)^2 + \sum |z^i|^2) + D(\psi - \psi_0)$). Then $\mu \geq 0$ in ∂V_δ and $\mu(p) = 0$. Moreover, we have

$$(4.23) \qquad \mathbf{L}(\sum |z^i|^2 + (r - 1)^2) = \sum_{i=1}^n (\tilde{g})^{i\bar{i}} + \frac{1}{2}(\tilde{g})^{n+1\overline{n+1}},$$

$$(4.24) \quad \begin{aligned} \mathbf{L}x^2 &= \frac{1}{2}\triangle_\psi x^2 \\ &= (\tilde{g})^{\alpha\bar{\beta}} X_\alpha X_{\bar{\beta}} x^2 + \triangle_\psi r \bar{X}_{n+1} x^2 \\ &= -2\sqrt{-1}x(\tilde{g})^{i\bar{j}} h_{i\bar{j}} + 2(\tilde{g})^{i\bar{j}} h_i h_{\bar{j}} - (\tilde{g})^{i\overline{n+1}} h_i r^{-1} - (\tilde{g})^{n+1\bar{j}} h_{\bar{j}} r^{-1} \\ &\quad + \frac{1}{2}(\tilde{g})^{n+1\overline{n+1}} r^{-2} - \frac{1}{2}\sqrt{-1}(\tilde{g})^{n+1\overline{n+1}} r^{-2} x + \sqrt{-1}(\triangle_\psi r) r^{-1} x \\ &= 2(\tilde{g})^{i\bar{j}} h_i h_{\bar{j}} - (\tilde{g})^{i\overline{n+1}} h_i r^{-1} - (\tilde{g})^{n+1\bar{j}} h_{\bar{j}} r^{-1} + \frac{1}{2}(\tilde{g})^{n+1\overline{n+1}} r^{-2} \\ &\leq 2(\sum_{i=1}^n |h_i|^2 + \frac{1}{4}r^{-2})(\sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}) \\ &\leq 3\sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}. \end{aligned}$$

Choosing d_1 large, by (4.9) and Lemma 4,

$$(4.25) \qquad \mathbf{L}\mu \leq (-\frac{a_0}{9}d_1 + 4d_2 + C_1)(1 + \sum_{\alpha=1}^{n+1}(\tilde{g})^{\alpha\bar{\alpha}}) < 0.$$

The Maximum principle implies that $\mu \geq 0$ in V_δ. Since $\mu(p) = 0$, we have $\frac{\partial \mu}{\partial r} \geq 0$ when $p = (q, 1)$ (or $\frac{\partial \mu}{\partial r} \leq 0$ when $p = (q, \frac{3}{2})$). In other word, we can choose a uniform constant C_5 which depending only on ψ_0, ρ and $\bar{g}$ such that

$$(4.26) \qquad -D\frac{\partial \psi}{\partial r}(p) \leq C_5 A.$$

Since D is any local first order constant differential operator, by replacing D with $-D$, we get

$$(4.27) \qquad D\frac{\partial \psi}{\partial r}(p) \leq C_5 A.$$

Therefore, we have

$$(4.28) \qquad |\frac{\partial^2 \psi}{\partial r \partial z^i}(p)| \leq C_5 A.$$

for a uniform constant C_5. $\square$

Proposition 4. *If ψ is a solution of equation (2.17) for $0 < \epsilon < 1$, then there exists a constant C_6 which depends only on $\rho, \psi_0, \|f^{\frac{1}{n}}\|_{C^{1,1}}$ and $(\overline{M}, \bar{g})$ such that for any unit vectors T_i, T_i on $\overline{M}$*

$$(4.29) \qquad \max_{\partial \overline{M}} |T_i T_j \psi| \leq C_6 \max_{M}(|d\psi|^2_{\bar{g}} + 1).$$

And specially

$$(4.30) \qquad \max_{\partial \overline{M}} |\triangle_{\bar{g}} \psi| \leq C_6 \max_{M}(|d\psi|^2_{\bar{g}} + 1).$$

Proof. We only need to get double normal derivative estimate. At point $p \in \partial\overline{M}$, choosing a local coordinates centered at p as above, equation (2.17) reduces to

$$(4.31) \qquad \det(\bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2} X_\alpha \bar{X}_\beta \psi) = 2^{-(n+1)} f r^{2n},$$

where $\bar{g}_{i\bar{j}} = \frac{1}{2} r^2 \delta_{ij}$, $\bar{g}_{n+1\overline{n+1}} = \frac{1}{2}$, $\bar{g}_{i\overline{n+1}} = \bar{g}_{n+1\bar{j}} = 0$. Denoting $E_{i\bar{j}} = \bar{g}_{i\bar{j}} + \frac{r^2}{2} X_i \bar{X}_j \psi_0$ and $E^{i\bar{k}} E_{j\bar{k}} = \delta_{ij}$. By the assumption (4.5) on the local coordinates, we conclude that $\frac{1}{4} a_0 \delta_{ij} \leq E_{i\bar{j}} \leq \frac{9}{4} a_0^{-1} \delta_{ij}$. Then,

$$(4.32) \begin{array}{rl} 0 & < \quad \frac{r^2}{8} \frac{\partial^2 \psi}{\partial r^2}(p) + \frac{1}{2} = det(E_{i\bar{j}})^{-1} 2^{-(n+1)} f r^{2n} + \frac{1}{16} \frac{\partial^2 \psi}{\partial z^i \partial r} E^{i\bar{j}} \frac{\partial^2 \psi}{\partial \bar{z}^j \partial r} \\ & \leq \quad 2^{n-1} a_0^{-n} f r^{2n} + 4 a_0^{-1}(\sum_{i=1}^{n} |\frac{\partial^2 \psi}{\partial z^i \partial r}|^2(p)). \end{array}$$

By Lemma 5, we may pick a uniform constant C_7 such that

$$(4.33) \qquad |\frac{\partial^2 \psi}{\partial r^2}(p)| \leq C_7 \max_{M}(|d\psi|^2_{\bar{g}} + 1).$$

$\square$

5. C^2_w ESTIMATE

We want to establish global C^2_w estimate in this section. For the standard complex Monge-Ampère equation on Kähler manifolds, C^2 a priori estimate was proved by Yau in [31] independent of the gradient estimate. For equation (2.17), the gradient estimate plays a crucial role. The global C^2_w estimate will depends on $\|f^{\frac{1}{n}}\|_{C^{1,1}}$. The gradient estimate on ψ depends on $\|f^{\frac{1}{n+1}}\|_{C^1}$. By (4.8), $\|f^{\frac{1}{n+1}}\|_{C^1} \leq C\|f^{\frac{1}{n}}\|_{C^{1,1}}$. Therefore, we will assume $\|\psi\|_{C^1}$ is bounded.

Since $\sqrt{-1}\partial\bar{\partial}r$ is a positive $(1,1)$ form, it determines a Kähler metric K on $\overline{M}$. Choose a local coordinates $(z^1, \cdots, z^n, w)$ as in (2.3) on $\overline{M}$, where $(x, z^1, \cdots, z^n)$ is a local Sasakian coordinates on M, and $\{X_\alpha\}_{\alpha=1}^{n+1}$, $\{\theta_\alpha\}_{\alpha=1}^{n+1}$ defined as in (2.4). It's easy to check that

$$(5.1) \begin{array}{rl} \sqrt{-1}\partial\bar{\partial}r & = \quad \frac{r}{2} d\eta + \frac{1}{2} dr \wedge \eta \\ & = \quad r^{-1}\bar{\omega} - \frac{1}{2} dr \wedge \eta \\ & = \quad \sqrt{-1} r h_{i\bar{j}} \theta^i \wedge \bar{\theta}^j + \sqrt{-1}(4r)^{-1} \theta^{n+1} \wedge \bar{\theta}^{n+1}, \end{array}$$

where $i, j = 1, \cdots, n$. Therefore,

$$(5.2) \qquad K = rg + (2r)^{-1} dr^2 - \frac{r}{2} \eta \otimes \eta = r^{-1}\bar{g} - (2r)^{-1} dr^2 - \frac{r}{2} \eta \otimes \eta,$$

and

$$(5.3) \qquad K_{i\bar{j}} = r h_{i\bar{j}}, \quad K_{i\overline{n+1}} = K_{n+1\bar{j}} = 0, \quad K_{n+1\overline{n+1}} = (4r)^{-1},$$

where $K_{\alpha\bar{\beta}} = <X_\alpha, \bar{X}_\beta>_K$. For any vector $Y = Y^\alpha X_\alpha + \bar{Y}^\beta \bar{X}_\beta$, we have

$$
\begin{aligned}
(5.4) \qquad <\frac{\partial}{\partial r}, Y>_K &= Y^\alpha <\frac{\partial}{\partial r}, X_\alpha>_K + \bar{Y}^\beta <\frac{\partial}{\partial r}, \bar{X}_\beta>_K \\
&= Y^{n+1} <\frac{\partial}{\partial r}, X_{n+1}>_K + \bar{Y}^{n+1} <\frac{\partial}{\partial r}, \bar{X}_{n+1}>_K \\
&= (4r)^{-1}(Y^{n+1} + \bar{Y}^{n+1}) = (2r)^{-1} dr(Y) \\
&= (2r)^{-1} <\nabla^K r, Y>_K,
\end{aligned}
$$

and

$$
(5.5) \qquad \frac{\partial}{\partial r} = (2r)^{-1} \nabla^K r,
$$

where $\nabla^k r$ is the gradient of r corresponding to the metric K.

Recall

$$
[X_i, \bar{X}_j] = -2\sqrt{-1} h_{i\bar{j}} \frac{\partial}{\partial x}, \quad [X_{n+1}, \bar{X}_{n+1}] = -\frac{1}{2}\sqrt{-1} r^{-1} \frac{\partial}{\partial x},
$$

$$
\nabla^K_{X_\alpha} \bar{X}_\beta - \nabla^K_{\bar{X}_\beta} X_\alpha = [X_\alpha, \bar{X}_\beta],
$$

$$
\nabla^K_{X_\alpha} \bar{X}_\beta + \nabla^K_{\bar{X}_\beta} X_\alpha = \sqrt{-1} J([X_\alpha, \bar{X}_\beta]).
$$

and

$$
\nabla^K_{X_\alpha} \bar{X}_\beta = \frac{1}{2}([X_\alpha, \bar{X}_\beta] + \sqrt{-1} J([X_\alpha, \bar{X}_\beta])).
$$

By above, give any smooth function φ on $\overline{M}$, we have

$$
\begin{aligned}
(5.6) \qquad \frac{1}{2}\triangle_K \varphi &= K^{\alpha\bar{\beta}} \nabla^K d\varphi(X_\alpha, \bar{X}_\beta) \\
&= K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta \varphi - K^{\alpha\bar{\beta}} d\varphi(\nabla^K_{X_\alpha} \bar{X}_\beta) \\
&= K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta \varphi + (n+1) d\varphi(\frac{\partial}{\partial r} + \sqrt{-1} r^{-1} \frac{\partial}{\partial x}),
\end{aligned}
$$

where $K^{\alpha\bar{\beta}}$ satisfies $K^{\alpha\bar{\beta}} K_{\gamma\bar{\beta}} = \delta_{\alpha\gamma}$.

We note that $\Delta_{\bar{g}} \psi$ and $\Delta_K \psi$ are equivalent as $\|\psi\|_{C^1}$ is bounded.

Lemma 6. *Let ψ be a smooth function on $\overline{M}$ and satisfy $\xi\psi \equiv 0$, then*

$$
(5.7) \qquad \triangle_K(\frac{\partial\psi}{\partial r}) = \frac{\partial}{\partial r}(\triangle_K \psi) + r^{-1}\triangle_K \psi - 2(n+1) r^{-1} \frac{\partial\psi}{\partial r} - 4\frac{\partial^2\psi}{\partial r^2}.
$$

Proof. It is straightforward to check that

$$
\begin{aligned}
(5.8) \qquad & K^{i\bar{j}} = r^{-1} h^{i\bar{j}}, \quad K^{n+1\overline{n+1}} = 4r, K^{i\overline{n+1}} = K^{n+1\bar{j}} = 0, \\
& \frac{\partial}{\partial r} X_i = X_i \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial r} \bar{X}_j = \bar{X}_j \frac{\partial}{\partial r}, \\
& \frac{\partial}{\partial x} X_\alpha = X_\alpha \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x} \bar{X}_\beta = \bar{X}_\beta \frac{\partial}{\partial x}, \\
& \frac{\partial}{\partial r} X_{n+1} = X_{n+1} \frac{\partial}{\partial r} + \sqrt{-1} r^{-2} \frac{\partial}{\partial x}, \\
& \frac{\partial}{\partial r} \bar{X}_{n+1} = \bar{X}_{n+1} \frac{\partial}{\partial r} + \sqrt{-1} r^{-2} \frac{\partial}{\partial x},
\end{aligned}
$$

and

$$
\begin{aligned}
(5.9) \qquad & \frac{\partial}{\partial r}(K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta \psi) = \frac{\partial}{\partial r}(K^{\alpha\bar{\beta}}) X_\alpha \bar{X}_\beta \psi + K^{\alpha\bar{\beta}} \frac{\partial}{\partial r}(X_\alpha \bar{X}_\beta \psi) \\
&= -r^{-2} h^{i\bar{j}} X_i \bar{X}_j \psi + 4 X_{n+1} \bar{X}_{n+1} \psi + K^{i\bar{j}} X_i \bar{X}_j(\frac{\partial\psi}{\partial r}) \\
& \quad + K^{n+1\overline{n+1}} \frac{\partial}{\partial r}(X_{n+1} \bar{X}_{n+1} \psi) \\
&= -r^{-1} K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta \psi + 2\frac{\partial^2\psi}{\partial r^2} + K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta(\frac{\partial\psi}{\partial r}),
\end{aligned}
$$

where the condition $\xi\psi \equiv 0$ has been used. Thus,

$$
\begin{aligned}
\frac{\partial}{\partial r}(\triangle_K \psi) &= 2\frac{\partial}{\partial r}(K^{\alpha\bar{\beta}} X_\alpha \bar{X}_\beta \psi) + 2(n+1)\frac{\partial^2\psi}{\partial r^2} \\
&= \triangle_K(\frac{\partial\psi}{\partial r}) - r^{-1}\triangle_K \psi + 2(n+1) r^{-1}\frac{\partial\psi}{\partial r} + 4\frac{\partial^2\psi}{\partial r^2}.
\end{aligned}
$$

□

Suppose ψ is a solution of equation (2.17) for some $0 < \epsilon < 1$ and Ω_ψ is positive. As above, let $\tilde{g}$ be the Hermitian metric induced by positive $(1,1)$ form Ω_ψ. From above,

$$\tilde{g}(X_\alpha, \bar{X}_\beta) = \bar{g}(X_\alpha, \bar{X}_\beta) + \frac{1}{2}r^2 X_\alpha \bar{X}_\beta \psi.$$

Thus,

$$
\begin{aligned}
\tfrac{1}{2}Tr_K\tilde{g} &= \tilde{g}(X_\alpha, \bar{X}_\beta)K^{\alpha\bar{\beta}} \\
(5.10) \quad &> \tilde{g}(X_{n+1}, \bar{X}_\beta)K^{n+1\bar{\beta}} = \tilde{g}(X_{n+1}, \bar{X}_{n+1})K^{n+1\overline{n+1}} \\
&= 4r(\bar{g}(X_{n+1}, \bar{X}_{n+1}) + \tfrac{1}{2}r^2 X_{n+1}\bar{X}_{n+1}\psi) = 2r + \tfrac{1}{2}r^2\frac{\partial^2\psi}{\partial r^2}.
\end{aligned}
$$

In what follows, the Kähler metric K will be considered as the background metric. Let p be a point of $\overline{M}$, choose a normal holomorphic local coordinates $(z^1, \cdots, z^{n+1})$ centered at p, and such that $K_{\alpha\bar{\beta}}(p) = \delta_{\alpha\delta}$, $dK_{\alpha\bar{\beta}}(p) = 0$. By the definition, $K_{\alpha\bar{\beta}} = r_{\alpha\bar{\beta}}$, and $\tilde{g}_{\alpha\bar{\beta}} = \bar{g}_{\alpha\bar{\beta}} + \frac{r^2}{2}\psi_{\alpha\bar{\beta}} - \frac{r^2}{2}\frac{\partial\psi}{\partial r}K_{\alpha\bar{\beta}}$. We may also assume that $\{\tilde{g}_{\alpha\bar{\beta}}\}$ is diagonal at the point p. For two fixed metric K and $\bar{g}$, there exist two positive constant d_1 and d_2 such that

$$(5.11) \qquad\qquad d_1\bar{g} \leq K \leq d_2\bar{g}.$$

By direct calculation,

$$
\begin{aligned}
0 \quad &< \quad 2r^{-2}Tr_K\tilde{g} = 2r^{-2}Tr_K\bar{g} + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r} \\
(5.12) \quad &\leq \quad r^{-2}\frac{4}{d_1}(n+1) + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r} \\
&\leq \quad \frac{4}{d_1}(n+1) + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r}
\end{aligned}
$$

Now, setting

$$(5.13) \qquad\qquad \zeta = 2 + \frac{4}{d_1}(n+1) + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r},$$

and

$$(5.14) \qquad\qquad u = \log\zeta + A_1|\partial\psi|_K^2 - A_2\psi,$$

where constants A_1 and A_2 are chosen sufficiently large. Denoting the Chern connection of the Hermitian metric $\tilde{g}$ by $\tilde{\nabla}$, and the canonical Laplacian corresponding with the connection $\tilde{\nabla}$ by $\tilde{\triangle}$.

Lemma 7. *There exist positive constants B_1, B_2, B_3 and B_4 depending only on r, $\max_{\overline{M}}|d\psi|_K^2$, $\|f^{\frac{1}{n}}\|_{C^{1,1}(\overline{M})}$, metric K and metric $\bar{g}$ such that*

$$
\begin{aligned}
\tfrac{1}{2}\tilde{\triangle}u \quad &\geq \quad -\tfrac{1}{2}A_2\tilde{\triangle}\psi - B_2 + Tr_{\tilde{g}}K[-B_1(1+\zeta^{\frac{-1}{n+1}}) - (n+3)\zeta^{-1}\frac{\partial^2\psi}{\partial r^2} \\
(5.15) \quad &\quad -A_1B_3 - B_4 + \tfrac{1}{2}\zeta^{-1}\frac{\partial}{\partial r}(\triangle_K\psi) + \tfrac{1}{2}A_1K^{\alpha\bar{\beta}}(\frac{\partial\psi_\alpha}{\partial r}\psi_{\bar{\beta}} + \frac{\partial\psi_{\bar{\beta}}}{\partial r}\psi_\alpha)] \\
&\quad +(A_1 - 4(n+1) - \tfrac{1}{2}n^2)\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2).
\end{aligned}
$$

Proof. With the local coordinates picked above,

$$
\begin{aligned}
\tfrac{1}{2}\tilde{\triangle}u &= \tfrac{1}{2}\tilde{\triangle}(\log\zeta + A_1|\partial\psi|_K^2 - A_2\psi) \\
(5.16) \quad &= \tilde{g}^{\gamma\bar{\delta}}(\log\zeta + A_1|\partial\psi|_K^2 - A_2\psi)_{\gamma\bar{\delta}} \\
&= \tilde{g}^{\gamma\bar{\delta}}(\log\zeta)_{\gamma\bar{\delta}} + A_1\tilde{g}^{\gamma\bar{\delta}}(K^{\alpha\bar{\beta}}\psi_\alpha\psi_{\bar{\beta}})_{\gamma\bar{\delta}} - A_2\tilde{g}^{\gamma\bar{\delta}}\psi_{\gamma\bar{\delta}} \\
&= \zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}\zeta_{\gamma\bar{\delta}} - \zeta^{-2}\tilde{g}^{\gamma\bar{\delta}}\zeta_\gamma\zeta_{\bar{\delta}} + A_1\tilde{g}^{\gamma\bar{\delta}}(K^{\alpha\bar{\beta}}\psi_\alpha\psi_{\bar{\beta}})_{\gamma\bar{\delta}} - A_2\tilde{g}^{\gamma\bar{\delta}}\psi_{\gamma\bar{\delta}}.
\end{aligned}
$$

At the point p,

$$(5.17) \quad \begin{aligned} \zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}\zeta_{\gamma\bar{\delta}} &= 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}(K^{\alpha\bar{\beta}}\psi_{\alpha\bar{\beta}} - (n+1)\frac{\partial\psi}{\partial r})_{\gamma\bar{\delta}} \\ &= 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}\{K^{\alpha\bar{\beta}}_{,\gamma\bar{\delta}}\psi_{\alpha\bar{\beta}} + K^{\alpha\bar{\beta}}\psi_{\alpha\bar{\beta}\gamma\bar{\delta}} - (n+1)(\frac{\partial\psi}{\partial r})_{\gamma\bar{\delta}}\}, \end{aligned}$$

$$(5.18) \quad \begin{aligned} & 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}\psi_{\alpha\bar{\beta}\gamma\bar{\delta}} \\ =\ & 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}\{2r^{-2}(\tilde{g}_{\gamma\bar{\delta}} - \bar{g}_{\gamma\bar{\delta}}) + \frac{\partial\psi}{\partial r}r_{\gamma\bar{\delta}}\}_{\alpha\bar{\beta}} \\ =\ & 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}\{12r^{-4}r_\alpha r_{\bar{\beta}}(\tilde{g}_{\gamma\bar{\delta}} - \bar{g}_{\gamma\bar{\delta}}) \\ & -4r^{-3}r_{\alpha\bar{\beta}}(\tilde{g}_{\gamma\bar{\delta}} - \bar{g}_{\gamma\bar{\delta}}) - 4r^{-3}r_\alpha(\tilde{g}_{\gamma\bar{\delta},\bar{\beta}} - \bar{g}_{\gamma\bar{\delta},\bar{\beta}}) \\ & -4r^{-3}r_{\bar{\beta}}(\tilde{g}_{\gamma\bar{\delta},\alpha} - \bar{g}_{\gamma\bar{\delta},\alpha}) + 2r^{-2}(\tilde{g}_{\gamma\bar{\delta},\alpha\bar{\beta}} - \bar{g}_{\gamma\bar{\delta},\alpha\bar{\beta}}) \\ & +r_{\gamma\bar{\delta}}(\frac{\partial\psi}{\partial r})_{\alpha\bar{\beta}} + \frac{\partial\psi}{\partial r}r_{\gamma\bar{\delta}\alpha\bar{\beta}}\}, \end{aligned}$$

By Lemma 6,

$$(5.19) \quad \begin{aligned} & 2\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}r_{\gamma\bar{\delta}}(\frac{\partial\psi}{\partial r})_{\alpha\bar{\beta}} = \tfrac{1}{2}\zeta^{-1}(Tr_{\tilde{g}}K)\triangle_K(\frac{\partial\psi}{\partial r}) \\ =\ & \tfrac{1}{2}\zeta^{-1}(Tr_{\tilde{g}}K)\{\frac{\partial}{\partial r}(\triangle_K\psi) + r^{-1}\triangle_K\psi - 2(n+1)r^{-1}\frac{\partial\psi}{\partial r} - 4\frac{\partial^2\psi}{\partial r^2}\}, \end{aligned}$$

It follows from equation (2.17),

$$(5.20) \quad \begin{aligned} & 4r^{-2}\zeta^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}\tilde{g}_{\gamma\bar{\delta},\alpha\bar{\beta}} \\ =\ & 4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}(\tilde{g}^{\gamma\bar{\delta}}\tilde{g}_{\gamma\bar{\delta},\alpha})_{\bar{\beta}} - 4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}(\tilde{g}^{\gamma\bar{\delta}})_{\bar{\beta}}\tilde{g}_{\gamma\bar{\delta},\alpha} \\ =\ & 4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}(f^{-1}f_\alpha + (\log\det(\bar{g}_{\gamma\bar{\delta}}))_\alpha)_{\bar{\beta}} \\ & +4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}\tilde{g}^{\gamma\bar{\eta}}\tilde{g}_{\tau\bar{\eta},\bar{\beta}}\tilde{g}^{\tau\bar{\delta}}\tilde{g}_{\gamma\bar{\delta},\alpha}, \end{aligned}$$

and

$$(5.21) \quad \begin{aligned} & A_1\tilde{g}^{\gamma\bar{\delta}}(K^{\alpha\bar{\beta}}\psi_\alpha\psi_{\bar{\beta}})_{\gamma\bar{\delta}} \\ =\ & A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}_{,\gamma\bar{\delta}}\psi_\alpha\psi_{\bar{\beta}} + A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(\psi_{\alpha\gamma}\psi_{\bar{\beta}\bar{\delta}} + \psi_{\alpha\bar{\delta}}\psi_{\bar{\beta}\gamma}) \\ & +A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(\psi_{\alpha\gamma\bar{\delta}}\psi_{\bar{\beta}} + \psi_\alpha\psi_{\bar{\beta}\gamma\bar{\delta}}) \\ =\ & A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}_{,\gamma\bar{\delta}}\psi_\alpha\psi_{\bar{\beta}} + A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(\psi_{\alpha\gamma}\psi_{\bar{\beta}\bar{\delta}} + \psi_{\alpha\bar{\delta}}\psi_{\bar{\beta}\gamma}) \\ & +A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(2r^{-2}(\tilde{g}_{\gamma\bar{\delta}} - \bar{g}_{\gamma\bar{\delta}}) + \frac{\partial\psi}{\partial r}r_{\gamma\bar{\delta}})_\alpha\psi_{\bar{\beta}} \\ & +A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(2r^{-2}(\tilde{g}_{\gamma\bar{\delta}} - \bar{g}_{\gamma\bar{\delta}}) + \frac{\partial\psi}{\partial r}r_{\gamma\bar{\delta}})_{\bar{\beta}}\psi_\alpha \\ =\ & A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}_{,\gamma\bar{\delta}}\psi_\alpha\psi_{\bar{\beta}} + A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(\psi_{\alpha\gamma}\psi_{\bar{\beta}\bar{\delta}} + \psi_{\alpha\bar{\delta}}\psi_{\bar{\beta}\gamma}) \\ & +\tfrac{1}{2}A_1(Tr_{\tilde{g}}K)K^{\alpha\bar{\beta}}((\frac{\partial\psi}{\partial r})_\alpha\psi_{\bar{\beta}} + (\frac{\partial\psi}{\partial r})_{\bar{\beta}}\psi_\alpha) \\ & -4(n+1)r^{-3}A_1K^{\alpha\bar{\beta}}(r_\alpha\psi_{\bar{\beta}} + \psi_\alpha r_{\bar{\beta}}) \\ & +2r^{-3}A_1(Tr_{\tilde{g}}\bar{g})K^{\alpha\bar{\beta}}(r_\alpha\psi_{\bar{\beta}} + \psi_\alpha r_{\bar{\beta}}) \\ & +2A_1r^{-1}K^{\alpha\bar{\beta}}\{[f^{-1}f_\alpha + (\log\det(\bar{g}_{\gamma\bar{\delta}}))_\alpha]\psi_{\bar{\beta}} \\ & +[f^{-1}f_{\bar{\beta}} + (\log\det(\bar{g}_{\gamma\bar{\delta}}))_{\bar{\beta}}]\psi_\alpha\} \\ & -2A_1r^{-1}\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}(\bar{g}_{\gamma\bar{\delta},\alpha}\psi_{\bar{\beta}} + \bar{g}_{\gamma\bar{\delta},\bar{\beta}}\psi_\alpha) \\ & +A_1\tilde{g}^{\gamma\bar{\delta}}K^{\alpha\bar{\beta}}\frac{\partial\psi}{\partial r}(r_{\gamma\bar{\delta}\alpha}\psi_{\bar{\beta}} + r_{\gamma\bar{\delta}\bar{\beta}}\psi_\alpha). \end{aligned}$$

Note that $Tr_{\tilde{g}}K \geq (Tr_K(\tilde{g}))^{\frac{1}{n+1}}f^{-\frac{1}{n}} \geq \frac{1}{2}\zeta^{\frac{1}{n+1}}f^{-\frac{1}{n}}$. Together with the assumption $f^{\frac{1}{n}} \in C^{1,1}$, we get

$$(5.22) \quad |(\frac{f_\alpha}{f})_{\bar{\beta}}(Z)| + |\frac{f_\alpha}{f}(Z)|^2 \leq Cf^{-\frac{1}{n}}(Z) \leq 2C\frac{Tr_{\tilde{g}}K(Z)}{\zeta^{\frac{1}{n+1}}(Z)}, \quad \forall Z \in \overline{M}.$$

On the other hand,

$$
\begin{aligned}
\zeta_\gamma &= (\triangle_K \psi - 2(n+1)\tfrac{\partial\psi}{\partial r})_\gamma \\
&= 2\{K^{\alpha\bar\beta}(\psi_{\alpha\bar\beta} - \tfrac{\partial\psi}{\partial r}r_{\alpha\bar\beta})\}_\gamma \\
&= 2\{K^{\alpha\bar\beta}(\psi_{\alpha\bar\beta\gamma} - \tfrac{\partial\psi}{\partial r}r_{\alpha\bar\beta\gamma}) - (n+1)(\tfrac{\partial\psi}{\partial r})_\gamma\} \\
(5.23)\quad &= 2\{K^{\alpha\bar\beta}(\psi_{\bar\beta\gamma} - \tfrac{\partial\psi}{\partial r}r_{\bar\beta\gamma})_\alpha + K^{\alpha\bar\beta}(\tfrac{\partial\psi}{\partial r})_\alpha r_{\gamma\bar\beta} - (n+1)(\tfrac{\partial\psi}{\partial r})_\gamma\} \\
&= 2\{K^{\alpha\bar\beta}(2r^{-2}(\tilde g_{\gamma\bar\beta} - \bar g_{\gamma\bar\beta}))_\alpha - n(\tfrac{\partial\psi}{\partial r})_\gamma\} \\
&= 4r^{-2}K^{\alpha\bar\beta}\tilde g_{\gamma\bar\beta,\alpha} - 4r^{-2}K^{\alpha\bar\beta}\bar g_{\gamma\bar\beta,\alpha} - 8r^{-3}K^{\alpha\bar\beta}\tilde g_{\gamma\bar\beta}r_\alpha \\
&\quad + 8r^{-3}K^{\alpha\bar\beta}\bar g_{\gamma\bar\beta}r_\alpha - n(\tfrac{\partial\psi}{\partial r}).
\end{aligned}
$$

By the Schwarz inequality, at point p,

$$
\begin{aligned}
-\zeta^{-2}\tilde g^{\gamma\bar\delta}\zeta_\gamma\zeta_{\bar\delta} &= -\zeta^{-2}\tilde g^{\gamma\bar\gamma}\zeta_\gamma\zeta_{\bar\gamma} \\
&\geq -16(1+\sigma)r^{-4}\zeta^{-2}(\sum_\gamma\tilde g^{\gamma\bar\gamma}|\sum_\alpha\tilde g_{\gamma\bar\alpha,\alpha}|^2) \\
(5.24)\quad &\quad -64(1+\sigma^{-1})r^{-4}\zeta^{-2}(\sum_\gamma\tilde g^{\gamma\bar\gamma}|\sum_\alpha\bar g_{\gamma\bar\alpha,\alpha}|^2) \\
&\quad -64(1+\sigma^{-1})r^{-6}\zeta^{-2}(\sum_\alpha\tilde g_{\alpha\bar\alpha}r_\alpha r_{\bar\alpha}) \\
&\quad -64(1+\sigma^{-1})r^{-6}\zeta^{-2}(\sum_\alpha(\sum_\gamma\tilde g^{\gamma\bar\gamma}|\sum_\alpha r_\alpha\bar g_{\gamma\bar\alpha}|^2) \\
&\quad -4(1+\sigma^{-1})n^2\zeta^{-2}(\sum_\gamma\tilde g^{\gamma\bar\gamma}(\tfrac{\partial\psi}{\partial r})_\gamma(\tfrac{\partial\psi}{\partial r})_{\bar\gamma}).
\end{aligned}
$$

and,

$$
\begin{aligned}
|\sum_\alpha\tilde g_{\gamma\bar\alpha,\alpha}|^2 &= |\sum_\alpha(\tfrac{1}{\sqrt{\tilde g_{\alpha\bar\alpha}}}\tilde g_{\gamma\bar\alpha,\alpha})\sqrt{\tilde g_{\alpha\bar\alpha}}|^2 \\
(5.25)\quad &\leq (\sum_\delta\tilde g^{\delta\bar\delta}|\tilde g_{\gamma\bar\delta,\delta}|^2)(\sum_\beta\tilde g_{\beta\bar\beta}) \\
&= \tfrac{1}{2}(Tr_K\tilde g)(\sum_\delta\tilde g^{\delta\bar\delta}|\tilde g_{\gamma\bar\delta,\delta}|^2).
\end{aligned}
$$

In turn,

$$
\begin{aligned}
&4r^{-2}\zeta^{-1}K^{\alpha\bar\beta}\tilde g^{\gamma\bar\eta}\tilde g_{\tau\bar\eta,\bar\beta}\tilde g^{\tau\bar\delta}\tilde g_{\gamma\bar\delta,\alpha} \\
&= 4r^{-2}\zeta^{-1}\sum_{\alpha,\gamma,\delta}\tilde g^{\gamma\bar\gamma}\tilde g^{\delta\bar\delta}|\tilde g_{\gamma\bar\delta,\alpha}|^2 \\
(5.26)\quad &= 4r^{-2}\zeta^{-1}\sum_{\gamma,\delta}\tilde g^{\gamma\bar\gamma}\tilde g^{\delta\bar\delta}|\tilde g_{\gamma\bar\delta,\delta}|^2 \\
&\geq 8r^{-2}(\zeta Tr_K\tilde g)^{-1}\sum_\gamma\tilde g^{\gamma\bar\gamma}|\sum_\alpha\tilde g_{\gamma\bar\alpha,\alpha}|^2 \\
&= 16r^{-4}\zeta^{-2}(1 + \tfrac{\zeta-2r^{-2}Tr_K\bar g}{2r^{-2}Tr_K\tilde g})\sum_\gamma\tilde g^{\gamma\bar\gamma}|\sum_\alpha\tilde g_{\gamma\bar\alpha,\alpha}|^2.
\end{aligned}
$$

In local holomorphic coordinates, from (5.5), we have

$$
(5.27)\qquad \frac{\partial}{\partial r} = (2r)^{-1}\nabla^K r = (2r)^{-1}(K^{\tau\bar\eta}\frac{\partial r}{\partial z^\tau}\frac{\partial}{\partial\bar z^\eta} + K^{\tau\bar\eta}\frac{\partial r}{\partial\bar z^\eta}\frac{\partial}{\partial z^\tau}).
$$

Thus,

$$
\begin{aligned}
(\tfrac{\partial\psi}{\partial r})_\gamma &= \tfrac{\partial}{\partial r}(\psi_\gamma) + ((2r)^{-1}K^{\tau\bar\eta}r_\tau)_\gamma\psi_{\bar\eta} + ((2r)^{-1}K^{\tau\bar\eta}r_{\bar\eta})_\gamma\psi_\tau \\
(5.28)\quad &= (2r)^{-1}(K^{\tau\bar\eta}r_\tau\psi_{\gamma\bar\eta} + K^{\tau\bar\eta}r_{\bar\eta}\psi_{\gamma\tau}) \\
&\quad + ((2r)^{-1}K^{\tau\bar\delta}r_\tau)_\gamma\psi_{\bar\eta} + ((2r)^{-1}K^{\tau\bar\eta}r_{\bar\eta})_\gamma\psi_\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
(\tfrac{\partial\psi}{\partial r})_{\gamma\bar\delta} &= \{\tfrac{\partial}{\partial r}(\psi_\gamma) + ((2r)^{-1}K^{\tau\bar\eta}r_\tau)_\gamma\psi_{\bar\eta} + ((2r)^{-1}K^{\tau\bar\eta}r_{e\bar t a})_\gamma\psi_\tau\}_{\bar\delta} \\
(5.29)\quad &= \tfrac{\partial}{\partial r}(\psi_{\gamma\bar\delta}) + ((2r)^{-1}K^{\tau\bar\eta}r_\tau)_{\gamma\bar\delta}\psi_{\bar\eta} + ((2r)^{-1}K^{\tau\bar\eta}r_{\bar\eta})_{\gamma\bar\delta}\psi_\tau \\
&\quad + ((2r)^{-1}K^{\tau\bar\eta}r_\tau)_\gamma\psi_{\bar\eta\bar\delta} + ((2r)^{-1}K^{\tau\bar\eta}r_{\bar\eta})_\gamma\psi_{\tau\bar\delta} \\
&\quad + ((2r)^{-1}K^{\tau\bar\eta}r_\tau)_{\bar\delta}\psi_{\gamma\bar\eta} + ((2r)^{-1}K^{\tau\bar\eta}r_{\bar\eta})_{\bar\delta}\psi_{\gamma\tau}.
\end{aligned}
$$

Combining (5.16)—(5.21), (5.28) and (5.29), we have

$$
\begin{aligned}
&\tfrac{1}{2}\tilde{\triangle} u\\
\geq\ &-\tfrac{1}{2}A_2\tilde{\triangle}\psi - B_1(1+\zeta^{\frac{-1}{n+1}})Tr_{\tilde{g}}K - B_2 - (n+3)\zeta^{-1}\tfrac{\partial^2\psi}{\partial r^2}Tr_{\tilde{g}}K\\
&-A_1 B_3 Tr_{\tilde{g}}K + \tfrac{1}{2}A_1(Tr_{\tilde{g}}K)K^{\alpha\bar{\beta}}[\tfrac{\partial\psi_\alpha}{\partial r}\psi_{\bar{\beta}} + \tfrac{\partial\psi_{\bar{\beta}}}{\partial r}\psi_\alpha]\\
&+\tfrac{1}{2}\zeta^{-1}(Tr_{\tilde{g}}K)\tfrac{\partial}{\partial r}(\triangle_K\psi) + 4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}\tilde{g}^{\gamma\bar{\eta}}\tilde{g}_{\tau\bar{\eta},\bar{\beta}}\tilde{g}^{\tau\bar{\delta}}\tilde{g}_{\gamma\bar{\delta},\alpha}\\
&-\zeta^{-2}\tilde{g}^{\gamma\bar{\delta}}\zeta_\gamma\zeta_{\bar{\delta}} + (A_1 - 4(n+1))\{\textstyle\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2)\},
\end{aligned}
\tag{5.30}
$$

where positive constants B_1, B_2, B_3 depend only on r, $\max_{\overline{M}}|d\psi|^2_K$, $\|f^{\frac{1}{n}}\|_{C^{1,1}}$, metric K and metric $\bar{g}$. From (5.24), (5.26) and (5.28) and, we can pick a constant B_4 depending only on r, metric K and metric $\bar{g}$, such that

$$
\begin{aligned}
&+4r^{-2}\zeta^{-1}K^{\alpha\bar{\beta}}\tilde{g}^{\gamma\bar{\eta}}\tilde{g}_{\tau\bar{\eta},\bar{\beta}}\tilde{g}^{\tau\bar{\delta}}\tilde{g}_{\gamma\bar{\delta},\alpha} - \zeta^{-2}\tilde{g}^{\gamma\bar{\delta}}\zeta_\gamma\zeta_{\bar{\delta}},\\
\geq\ &-B_4 Tr_{\tilde{g}}K - \tfrac{1}{2}n^2\{\textstyle\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2)\}.
\end{aligned}
\tag{5.31}
$$

The lemma now follows from (5.30) and (5.31). $\qquad\square$

We are ready to prove the following estimate.

Proposition 5. *Let ψ be a solution of (2.17) for some $0 < \epsilon \leq 1$ with $\Omega_\psi > 0$. Let ζ be defined as in (5.13). There exist constants A_1, A_2 and A_3 which depend only on r, $\|f^{\frac{1}{n}}\|_{C^{1,1}(\overline{M})}$, $\max_{\overline{M}}|\psi|$, $\max_{\overline{M}}|d\psi|^2_K$, metric K and metric $\bar{g}$, if the maximum value of u defined in (5.14) is achieved at an interior point p, then $u(p) \leq A_3$.*

As a consequence, for any $0 < f \in C_B^\infty(\overline{M})$ and basic boundary value ψ_0, there exists constant C depending only on $\|f\|_{C^{1,1}(\overline{M})}$, $\|\psi_0\|_{C^{2,1}}$, and metric $\bar{g}$, such that

$$
\|\psi\|_{C^2_w} \leq C.
\tag{5.32}
$$

Proof. Since p is an interior maximum point of u, at p point,

$$
\begin{aligned}
0\ =\ &\tfrac{\partial u}{\partial r}\\
=\ &\zeta^{-1}\tfrac{\partial\zeta}{\partial r} + A_1\tfrac{\partial}{\partial r}(|\partial\psi|^2_K) - A_2\tfrac{\partial\psi}{\partial r}\\
=\ &\zeta^{-1}\tfrac{\partial}{\partial r}(\triangle_K\psi) + A_1 K^{\alpha\bar{\beta}}[\tfrac{\partial\psi_\alpha}{\partial r}\psi_{\bar{\beta}} + \tfrac{\partial\psi_{\bar{\beta}}}{\partial r}\psi_\alpha] - A_2\tfrac{\partial\psi}{\partial r}\\
&-2(n+1)\tfrac{\partial^2\psi}{\partial r^2}\zeta^{-1}.
\end{aligned}
\tag{5.33}
$$

By (5.10),

$$
\zeta \geq 2 + 2r^{-2}Tr_K\tilde{g} > 2\frac{\partial^2\psi}{\partial r^2} + 2.
\tag{5.34}
$$

From (5.15), at point p,

$$
\begin{aligned}
0\ \geq\ &\tfrac{1}{2}\tilde{\triangle}u\\
\geq\ &A_2 r^{-2}Tr_{\tilde{g}}\bar{g} - 2(n+1)A_2 r^{-2} - \tfrac{1}{2}A_2\tfrac{\partial\psi}{\partial r}Tr_{\tilde{g}}K\\
&-(n+3)\zeta^{-1}\tfrac{\partial^2\psi}{\partial r^2}Tr_{\tilde{g}}K\\
&+\tfrac{1}{2}\zeta^{-1}(Tr_{\tilde{g}}K)\tfrac{\partial}{\partial r}(\triangle_K\psi) + \tfrac{1}{2}A_1(Tr_{\tilde{g}}K)K^{\alpha\bar{\beta}}[\tfrac{\partial\psi_\alpha}{\partial r}\psi_{\bar{\beta}} + \tfrac{\partial\psi_{\bar{\beta}}}{\partial r}\psi_\alpha]\\
&+(A_1 - 4(n+1) - \tfrac{1}{2}n^2)\{\textstyle\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2)\}\\
&-(B_1 + B_4)Tr_{\tilde{g}}K - B_2 - A_1 B_3 Tr_{\tilde{g}}K\\
\geq\ &\tfrac{4}{9d_2}A_2 Tr_{\tilde{g}}K - 2(n+1)A_2 - 2\zeta^{-1}\tfrac{\partial^2\psi}{\partial r^2}Tr_{\tilde{g}}K\\
&+(A_1 - 4(n+1) - \tfrac{1}{2}n^2)\{\textstyle\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2)\}\\
&-(B_1 + B_4)Tr_{\tilde{g}}K - B_2 - A_1 B_3 Tr_{\tilde{g}}K\\
\geq\ &(\tfrac{4}{9d_2}A_2 - 1 - B_1 - B_4 - A_1 B_3)Tr_{\tilde{g}}K - B_2 - 2(n+1)A_2\\
&+(A_1 - 4(n+1) - \tfrac{1}{2}n^2)\{\textstyle\sum_{\alpha,\gamma}(\tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\bar{\gamma}}|\psi_{\alpha\bar{\gamma}}|^2)\}.
\end{aligned}
\tag{5.35}
$$

Pick $A_1 = 4(n+1) + \frac{1}{2}n^2$, $A_2 = \frac{9}{4}(2 + B_1 + B_4 + A_1 B_3)d_2$, the above inequality yields at point p,

$$(5.36) \qquad\qquad Tr_{\tilde{g}}K \leq B_2 + 2(n+1)A_2.$$

On the other hand,

$$(5.37) \qquad \begin{aligned} (\tfrac{1}{2}Tr_{\tilde{g}}K)^n &\geq \tfrac{1}{2}(Tr_K\tilde{g})\frac{\det(K_{\alpha\bar{\beta}})}{\det(\tilde{g}_{\alpha\bar{\beta}})} \\ &= \tfrac{1}{2}(Tr_K\tilde{g})\frac{\det(K_{\alpha\bar{\beta}})}{f\det(\bar{g}_{\alpha\bar{\beta}})} \\ &\geq \tfrac{1}{2}(Tr_K\tilde{g})f^{-1}(d_1)^{n+1}, \end{aligned}$$

and

$$(5.38) \qquad \begin{aligned} 2Tr_K\tilde{g} &\geq 2r^{-2}Tr_K\tilde{g} \\ &= 2r^{-2}Tr_K\bar{g} + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r} \\ &\geq 4r^{-2}\tfrac{1}{d_2}(n+1) + \triangle_K\psi - 2(n+1)\frac{\partial\psi}{\partial r} \\ &= \zeta - 2 - \tfrac{4}{d_1}(n+1) + 4r^{-2}\tfrac{1}{d_2}(n+1) \\ &\geq \zeta - 2 - \tfrac{4}{d_1}(n+1) + \tfrac{16}{9d_2}(n+1). \end{aligned}$$

Since we already have estimated $|\psi|_{C^1}$, $|\triangle_{\bar{g}}\psi|$, $Tr_{\tilde{g}}\tilde{g}$ and $|\triangle_K\psi|$ are all equivalent. C_w^2 bound follows directly. The proof is complete. $\square$

We have established C_w^2 bound for any smooth solution ψ to the equation (2.17). For each For $f > 0$, equation (2.17) is strictly elliptic and concave. From this point, the theory of Evans and Krylov can be applied. In fact, with sufficient smooth boundary data, for a uniformly elliptic and concave fully nonlinear equation, the assumption of $u \in C^{1,\gamma}$ for some $\gamma > 0$ is suffice to get global $C^{2,\alpha}$ regularity (e.g., see Theorem 7.3 in [7]). The higher follows from the standard elliptic theory. By Lemma 2, the kernel of the linearized operator of (2.17) with null boundary data is trivial. The linearized equation is solvable by the Fredholm alternative. Theorem 2 is proved following the method of continuity.

As a consequence of Theorem 2, we obtain the first part of Theorem 1. We discuss the uniqueness of C_w^2 solutions of the Dirichlet problem (1.10) and prove the second part of Theorem 1.

Lemma 8. *Suppose ψ is a C_w^2 function defined on $\overline{M}$ with $\Omega_\psi \geq 0$ defined in (1.9). For any $\delta > 0$, there is a function $\psi_\delta \in C^\infty(\overline{M})$ such that $\delta\bar{\omega} \geq \Omega_{\psi_\delta} > 0$ and $\|\psi - \psi_\delta\|_{C_w^2}(\overline{M}) \leq \delta$, where $\bar{\omega}$ is the Kähler form on $\bar{M}$ and $\|.\|_{C_w^2}(\overline{M})$ is defined as in (1.13).*

Proof. $\psi \in \mathcal{W}\text{-}C^2$ implies that Ω_ψ is bounded (as $\|.\|_{C_w^2}$ controls the complex hessian). For any $\epsilon > 0$, set $\psi_\epsilon = (1-\epsilon)\psi + \epsilon r$ where r is a radial function in the Kähler cone $\bar{M}$. It is obvious $\Omega_{\psi_\epsilon} > 0$ and it is also bounded. We now approximate ψ_ϵ by a smooth function ψ_δ such that $\|\psi_\epsilon - \psi_\delta\|_{C_w^2(\bar{M})} \leq \epsilon^2$. It is clear that we can make $\Omega_{\psi_\delta} > 0$ and $|\Omega_{\psi_\delta} - \Omega_\psi|$ as small as we wish by shrinking ϵ. $\square$

Lemma 9. *C_w^2 solutions to the degenerate Monge-Ampére equation (1.10) with given boundary data are unique.*

Proof. Suppose there are two such solutions ψ_1, ψ_2 with the same boundary data. For any $0 < \delta < 1$, pick any $0 < \delta_1, \delta_2 < \delta$, by Lemma 8, there exist two smooth functions ψ_1' and ψ_2' such that

$$\Omega_{\psi_i'}^{n+1} = f_i\bar{\omega}^{n+1}$$

in $\overline{M}$, $\max_{\overline{M}} |\psi_i' - \psi_i| \le \delta_i$ and $0 < f_i < \delta_i$ for $i = 1, 2$. Set $\tilde{\psi}_1' = (1 - \delta)\psi_1' + \delta r$, where r is the radial function on $\bar{M}$. Since $\Omega_{\tilde{\psi}_1'}^{n+1} \ge \delta^{n+1} \bar{\omega}^{n+1}$ and $\Omega_{\psi_2}^{n+1} = 0, a.e.$, we may choose δ_2 sufficient small such that $0 < f_2 \bar{\omega}^{n+1} \le \Omega_{\tilde{\psi}_1'}^{n+1}$. The maximum principle implies $\max_{\overline{M}}(\tilde{\psi}_1' - \psi_2') \le \max_{\partial \overline{M}}(\tilde{\psi}_1' - \psi_2')$. Thus

$$\max_{\overline{M}}(\psi_1 - \psi_2) \le \max_{\partial \overline{M}}(\psi_1 - \psi_2) + C\delta = C\delta,$$

where constant C depends only on C^0 norm of ψ_1 and ψ_2. Interchange the role of ψ_1 and ψ_2, we have

$$\max_{\overline{M}} |\psi_1 - \psi_2| \le C\delta.$$

Since $0 < \delta < 1$ is arbitrary, we conclude that $\psi_1 = \psi_2$. $\square$

The proof of Theorem 1 is complete.

Remark 2. *One may deal with geodesic equation (1.7) in the setting of transverse Kähler geometry. Complexifying time variable t as in [22, 29, 10], one arrives a homogeneous complex Monge-Ampère equation in transverse Kähler setting. There is no problem to carry out interior estimates for this type of equation as in Kähler case [5]. But there is difficulty to prove the boundary regularity estimates including the direct gradient estimates, as the linearization of the equation is not elliptic (missing ξ direction). One needs to add term like ξ^2 to make it elliptic, this will cause other complications as well. Our approach via equation (2.17) put the problem in the frame of elliptic complex Monge-Ampère. The analysis developed here should be useful to deal complex Monge-Ampère type equations in other contexts.*

6. Applications

As in the case of the space of Kähler metrics [5], the regularity result of the geodesic equation has geometric implications on the Sasakian manifold (M, g). One of them is the uniqueness of transverse Kähler metric with constant scalar curvature in the given basic Kähler class. The discussions here are parallel to [5]. The proofs can be found in the Appendix.

Let us recall the definition of the natural connection on the space $\mathcal{H}$ in [19].

Definition 2. *Let $\varphi(t) : [0, 1] \to \mathcal{H}$ be any path in $\mathcal{H}$ and let $\psi(t)$ be another* **basic** *function on $M \times [0, 1]$, which we regard as a vector field along the path $\varphi(t)$. Define the covariant derivative of ψ along the path φ by*

$$(6.1) \qquad D_{\dot\varphi}\psi = \frac{\partial \psi}{\partial t} - \frac{1}{4} < d_B\psi, d_B\dot\varphi >_{g_\varphi},$$

where $<>_{g_\varphi}$ is the Riemannian inner product on co-tangent vectors to (M, g_φ), and $\dot\varphi = \frac{\partial \varphi}{\partial t}$.

The geodesic equation (1.7) can be written as

$$(6.2) \qquad\qquad\qquad D_{\dot\varphi}\dot\varphi = 0.$$

In [19], we have shown that: the connection D is compatible with the Weil-Peterson metric structure and torsion free; the sectional curvature of D is formally non-positive, $\mathcal{H}_0 \subset \mathcal{H}$ is totally geodesic and totally convex.

Let $\mathcal{K}$ be the space of all transverse Kähler form in the basic $(1,1)$ class $[d\eta]_B$, the natural map

$$(6.3) \qquad \mathcal{H} \to \mathcal{K}, \quad \varphi \mapsto \frac{1}{2}(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi)$$

is surjective. Normalize $\int_M \eta \wedge (d\eta)^n = 1$. Define a function $\mathcal{I} : \mathcal{H} \to R$ by

$$(6.4) \quad \mathcal{I}(\varphi) = \sum_{p=0}^{n} \frac{n!}{(p+1)!(n-p)!} \int_M \varphi\eta \wedge (d\eta)^{n-p} \wedge (\sqrt{-1}\partial_B\bar{\partial}_B\varphi)^p \, ,$$

Set

$$(6.5) \qquad \mathcal{H}_0 = \{\varphi \in \mathcal{H} | \mathcal{I}(\varphi) = 0\},$$

then

$$(6.6) \qquad \mathcal{H}_0 \cong \mathcal{K}, \quad \varphi \leftrightarrow \frac{1}{2}(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi),$$

and

$$(6.7) \qquad \mathcal{H} \cong \mathcal{H}_0 \times \mathbb{R}, \quad \varphi \leftrightarrow (\varphi - \mathcal{I}(\varphi), \mathcal{I}(\varphi)).$$

Recall for a given Sasakian structure (ξ, η, Φ, g), the exact sequence of vector bundles,

$$(6.8) \qquad 0 \to L\xi \to TM \to \nu(\mathcal{F}_\xi) \to 0,$$

generates the Reeb foliation $\mathcal{F}_\xi$ (where $L\xi$ is the trivial line bundle generated by the Reeb field ξ and $\nu(\mathcal{F}_\xi)$ is the normal bundle of the foliation $\mathcal{F}_\xi$). The metric g gives a bundle isomorphism $\sigma_g : \nu(\mathcal{F}_\xi) \to \mathcal{D}$, where $\mathcal{D} = ker\{\eta\}$ is the contact sub-bundle. $\Phi|_\mathcal{D}$ induces a complex structure $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.

$(\mathcal{D}, \Phi|_\mathcal{D}, d\eta)$ give M a transverse Kähler structure with transverse Kähler form $\frac{1}{2}d\eta$ and transverse metric g^T defined by

$$(6.9) \qquad g^T(\cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \Phi\cdot)$$

which is relate to the Sasakian metric g by

$$(6.10) \qquad g = g^T + \eta \otimes \eta.$$

For simplicity, the bundle metric $\sigma^* g^T$ still denoted by g^T. We will identify $\nu(\mathcal{F}_\xi)$ and $\mathcal{D}$ and $\sigma_g = id$ if there is no confusion. The transverse metric g^T induces a transverse Levi-Civita connection on $\nu(\mathcal{F}_\xi)$ by

$$(6.11) \qquad \nabla^T_X Y = \begin{cases} (\nabla_X Y)^p, & X \in \mathcal{D}, \\ [\xi, Y]^p, & X = \xi, \end{cases}$$

where Y is a section of $\mathcal{D}$ and X^p the projection of X onto $\mathcal{D}$, ∇ is the Levi-Civita connection of metric g. It is easy to check that the connection satisfies

$$\nabla^T_X Y - \nabla^T_Y X - [X, Y]^p = 0, \quad Xg^T(Z, W) = g^T(\nabla^T_X Z, W) + g^T(Z, \nabla^T_X W),$$

$\forall X, Y \in TM, Z, W \in \mathcal{D}$. This means that the transverse Levi-Civita connection is torsion-free and metric compatible. The transverse curvature relating with the above transverse connection is defined by

$$(6.12) \qquad R^T(V, W)Z = \nabla^T_V \nabla^T_W Z - \nabla^T_W \nabla^T_V Z - \nabla^T_{[V,W]}Z,$$

where $V, W \in TM$ and $Z \in \mathcal{D}$. The transverse Ricci curvature is defined as

$$(6.13) \qquad Ric^T(X, Y) = < \mathbb{R}^T(X, e_i)e_i, Y >_g,$$

where e_i is an orthonormal basis of $\mathcal{D}$ and $X, Y \in \mathcal{D}$. The following is held

$$(6.14) \qquad Ric^T(X, Y) = Ric(X, Y) + 2g^T(X, Y), \quad X, Y \in \mathcal{D}.$$

A Sasakian metric g is said to be η-Einstein if g satisfies

$$(6.15) \qquad\qquad Ric_g = \lambda g + \nu \eta \otimes \eta,$$

for some constants $\lambda, \nu \in R$. It is equivalent to be transverse Einstein in the sense that

$$(6.16) \qquad\qquad Ric^T = cg^T,$$

for certain constant c. The trace of transverse Ricci tensor is called the transverse scalar curvature, and which will be denoted by S^T.

Let $\rho^T(\cdot, \cdot) = Ric^T(\Phi\cdot, \cdot)$ and $\rho = Ric^T(\Phi\cdot, \cdot)$, ρ^T is called the transverse Ricci form. They satisfy the relation

$$(6.17) \qquad\qquad \rho^T = \rho + d\eta.$$

ρ^T is a closed basic $(1,1)$ form and the basic cohomology class $[\frac{1}{2\pi}\rho^T]_B = C_1^B(M)$ is the basic first Chern class. The basic first Chern class of M is called positive (resp. negative, null) if $C_1^B(M)$ contains a positive (resp. negative, null) representation, this condition is expressed by $C_1^B(M) > 0$ (resp. $C_1^B(M) < 0$, $C_1^B(M) = 0$).

Definition 3. *A complex vector field X on a Sasakian manifold (M, ξ, η, Φ, g) is called a* **transverse holomorphic vector field** *if it satisfies:*

(1) $\pi[\xi, X] = 0$;
(2) $\bar{J}(\pi(X)) = \sqrt{-1}\pi(X)$;
(3) $\pi([Y, X]) - \sqrt{-1}\bar{J}\pi([Y, X]) = 0$, $\forall Y$ *satisfying* $\bar{J}\pi(Y) = -\sqrt{-1}\pi(Y)$.

Let ψ be a basic function, then there is an unique vector field $V_\eta(\psi) \in \Gamma(T^c M)$ satisfies: (1) $\psi = \sqrt{-1}\eta(V_\eta(\psi))$; (2) $\bar{\partial}_B \psi = -\frac{\sqrt{-1}}{2}d\eta(V_\eta(\psi), \cdot)$. The vector field $V_\eta(\psi)$ is called the **Hamiltonian vector field** *of ψ corresponding to the Sasakian structure (ξ, η, Φ, g).*

With the local coordinate chart and the function h chosen as in (2.5), the transverse Ricci form can be expressed by

$$\rho^T = -\sqrt{-1}\partial_B\bar{\partial}_B \log\det(g_{i\bar{j}}^T) == -\sqrt{-1}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log\det(h_{k\bar{l}}))dz^i \wedge d\bar{z}^j.$$

In this setting, $\forall \varphi \in \mathcal{H}$, we have

$$(6.18) \quad
\begin{aligned}
\eta_\varphi &= dx - \sqrt{-1}((h_j + \tfrac{1}{2}\varphi_j)dz^j - (h_{\bar{j}} + \tfrac{1}{2}\varphi_{\bar{j}})d\bar{z}^j); \\
\Phi_\varphi &= \sqrt{-1}\{Y_j \otimes dz^j - \bar{Y}_j \otimes d\bar{z}^j\}; \\
g_\varphi &= \eta \otimes \eta + 2(h + \tfrac{1}{2}\varphi)_{i\bar{j}}dz^i d\bar{z}^j \\
d\eta_\varphi &= 2\sqrt{-1}(h + \tfrac{1}{2}\varphi)_{i\bar{j}}dz^i \wedge d\bar{z}^j, \\
g_\varphi^T &= 2(h + \tfrac{1}{2}\varphi)_{i\bar{j}}dz^i d\bar{z}^j, \\
\rho_\varphi^T &= -\sqrt{-1}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log\det((h + \tfrac{1}{2}\varphi)_{k\bar{l}}))dz^i \wedge d\bar{z}^j.
\end{aligned}$$

where $Y_j = \frac{\partial}{\partial z^j} + \sqrt{-1}((h_j + \tfrac{1}{2}\varphi_j)\frac{\partial}{\partial x}$ and $\bar{Y}_j = \frac{\partial}{\partial \bar{z}^j} - \sqrt{-1}((h_{\bar{j}} + \tfrac{1}{2}\varphi_{\bar{j}})\frac{\partial}{\partial x}$.

Remark 3. *A complex vector field X on the Sasakian manifold (M, ξ, η, Φ, g) is transverse holomorphic if and only if it satisfies:*

(1) $\Phi(X - \eta(X)\xi) = \sqrt{-1}(X - \eta(X)\xi)$,
(2) $[X, \xi] = \eta([X, \xi])\xi$ *or equivalently* $\nabla_\xi^T(X - \eta(X)\xi) = 0$;

(3) $\nabla^T_{Y-\eta(Y)\xi}(X - \eta(X)\xi) = 0$, $\forall Y$ satisfying $Y - \eta(Y)\xi \in \mathcal{D}^{0,1}$.

In local coordinates $(x, z^1, \cdots, z^n)$ as in (2.3), the transverse holomorphic vector field X can be written as

$$
\begin{aligned}
X &= \eta(X)\frac{\partial}{\partial x} + \sum_{i=1}^n X^i(\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\frac{\partial}{\partial x}) \\
&= \eta_\varphi(X)\frac{\partial}{\partial x} + \sum_{i=1}^n X^i(\frac{\partial}{\partial z^i} - \eta_\varphi(\frac{\partial}{\partial z^i})\frac{\partial}{\partial x})
\end{aligned}
$$

where X^i are local holomorphic basic functions, and $\varphi \in \mathcal{H}$.

The Hamiltonian vector field $V_{\eta_\varphi}(\psi)$ of the basic function ψ with respect to the Sasakian structure $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ can be written as

$$
V_{\eta_\varphi}(\psi) = -\sqrt{-1}\psi\frac{\partial}{\partial x} + \sum_{i=1}^n h^{i\bar{j}}_\varphi \frac{\partial\psi}{\partial \bar{z}^j}(\frac{\partial}{\partial z^i} - \eta_\varphi(\frac{\partial}{\partial z^i})\frac{\partial}{\partial x}),
$$

where $h^{i\bar{j}}_\varphi(h_\varphi)_{k\bar{j}} = \delta^i_k$, $(h_\varphi)_{k\bar{j}} = h_{k\bar{j}} + \frac{1}{2}\varphi_{k\bar{j}}$ and $\varphi \in \mathcal{H}$. In general, $V_{\eta_\varphi}(\psi)$ is not transversally holomorphic. If define $\bar{\partial}_B V_{\eta_\varphi}(\psi) \in \Gamma(\wedge^{0,1}_B(M) \otimes (\nu\mathcal{F}_\xi)^{1,0})$ by

$$
\bar{\partial}_B V_{\eta_\varphi}(\psi) = (h^{i\bar{j}}_\varphi \psi_{\bar{j}})_{\bar{k}} d\bar{z}^k \otimes \frac{\partial}{\partial z^i},
$$

$V_\eta(\psi)$ is transversally holomorphic if and only if $\bar{\partial}_B V_\eta(\psi) = 0$. In local coordinates (2.3), it is equivalent to

$$
\frac{\partial}{\partial \bar{z}^k}(h^{i\bar{j}}\frac{\partial\psi}{\partial \bar{z}^j}) = 0, \quad \forall i, k.
$$

Lemma 10. Let (M, ξ, η, Φ, g) be a Sasakian manifold and ψ be a real basic function on M. Assuming that $V_{\eta_\varphi}(\psi)$ is transverse holomorphic for some $\varphi \in \mathcal{H}$, where $V_{\eta_\varphi}(\psi)$ is the Hamiltonian vector field of ψ corresponding with the Sasakian structure $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$. If the basic first Chern lass $C^B_1(M) \leq 0$, then ψ must be a constant.

$\forall \varphi \in \mathcal{H}$, $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ defined in (1.3) and (1.4) is also a Sasakian structure on M. $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ and (ξ, η, Φ, g) have the same transversely holomorphic structure on $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$, and their transverse Kähler forms are in the same basic $(1, 1)$ class $[d\eta]_B$ (Proposition 4.2 in [14]). This class is called the **basic Kähler class** of the Sasakian manifold (M, ξ, η, Φ, g). All these Sasakian metrics have the same volume, as $\int_M \eta_\varphi \wedge (d\eta_\varphi)^n = \int_M \eta \wedge (d\eta)^n = 1$ (e.g., section 7 of [1]).

Let ρ^T_φ denote the transverse Ricci form of the Sasakian structure $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$. $\int_M \rho^T_\varphi \wedge (d\eta_\varphi)^2 \wedge \eta_\varphi$ is independent of the choice of $\varphi \in \mathcal{H}$ (e.g., Proposition 4.4 [14]). This means that

(6.19) $$\bar{S} = \frac{\int_M S^T_\varphi(d\eta_\varphi)^n \wedge \eta_\varphi}{\int_M (d\eta_\varphi)^n \wedge \eta_\varphi} = \frac{\int_M 2n\rho^T_\varphi \wedge (d\eta_\varphi)^{n-1} \wedge \eta}{\int_M (d\eta_\varphi)^n \wedge \eta},$$

depends only on the basic Kähler class. As in the Kähler case (see [21]), we have the following lemma.

Lemma 11. *Let φ' and φ'' are two basic functions in $\mathcal{H}$ and φ_t ($t \in [a, b]$) be a path in $\mathcal{H}$ connecting φ' and φ''. Then*

$$(6.20) \qquad \mathcal{M}(\varphi', \varphi'') = -\int_a^b \int_M \dot{\varphi}_t (S_t^T - \bar{S})(d\eta_t)^n \wedge \eta_t \, dt$$

is independent of the path φ_t, where $\dot{\varphi}_t = \frac{\partial}{\partial t}\varphi_t$, S_t^T is the transverse scalar curvature to the Sasakian structure $(\xi, \eta_t, \Phi_t, g_t)$ and $\bar{S}$ is the average defined as in (6.19) . Furthermore, $\mathcal{M}$ satisfies the 1-cocycle condition and

$$(6.21) \qquad \mathcal{M}(\varphi' + C', \varphi'' + C'') = \mathcal{M}(\varphi', \varphi'')$$

for any $C', C'' \in \mathbb{R}$.

In view of (6.21), $\mathcal{M} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ factors through $\mathcal{H}_0 \times \mathcal{H}_0$. Hence we can define the mapping $\mathcal{M} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ by the identity $\mathcal{K} \cong \mathcal{H}_0$

$$(6.22) \qquad \mathcal{M}(d\eta', d\eta'') := \mathcal{M}(\varphi', \varphi''),$$

where $d\eta', d\eta'' \in \mathcal{K}$, and $\varphi', \varphi'' \in \mathcal{H}$ such that $d\eta' = d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \varphi'$ and $d\eta'' = d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \varphi''$.

Definition 4. *The mapping*

$$(6.23) \qquad \mu : \mathcal{K} \to \mathbb{R}, \quad d\eta^* \mapsto \mu(d\eta^*) := \mathcal{M}(d\eta, d\eta^*)$$

is called the $\mathcal{K}$-energy map of the transverse Kähler class in $[d\eta]_B$. The mapping $\mu : \mathcal{H} \to \mathbb{R}$, $\mu(\varphi) := \mathcal{M}(0, \varphi)$ is also called by the $\mathcal{K}$-energy map of the space $\mathcal{H}$.

Lemma 12. *For every smooth path $\{\varphi_t | a \le t \le b\}$, we have*

$$(6.24) \qquad \begin{aligned} \frac{d^2}{dt^2}\mu(\varphi_t) = {} & -(D_{\dot{\varphi}_t}\dot{\varphi}_t, S_t^T - \bar{S})_{\varphi_t} \\ & + \int_M \frac{1}{2}|\bar{\partial}_B V_{\varphi_t}(\dot{\varphi}_t)|^2_{\varphi_t} (d\eta_t)^n \wedge \eta_t. \end{aligned}$$

where $\bar{\partial}_B V_{\varphi_t}(\dot{\varphi}_t) = (h_t^{i\bar{j}}(\dot{\varphi}_t)_{\bar{j}})_{\bar{k}} d\bar{z}^k \otimes \frac{\partial}{\partial z^i}$ in local coordinates. $\mu : \mathcal{H} \to \mathbb{R}$ is a convex function, i.e. the Hessian of μ is nonnegative everywhere on $\mathcal{H}$.

Define

$$(6.25) \qquad \bar{\mathcal{H}} = \{\text{completion of } \mathcal{H} \text{ under the norm } \|.\|_{C_w^2}.\}$$

Let φ_0, φ_1 be two points in $\mathcal{H}$, by Theorem 1, there exists an unique $\mathcal{W}$-C^2 geodesic $\varphi_t : [0, 1] \to \bar{\mathcal{H}}$ connecting them.

Definition 5. *Let φ_0, φ_1 be two points in $\mathcal{H}$, and $\varphi_t : [0, 1] \to \bar{\mathcal{H}}$ be the C_w^2 geodesic connecting these two points. The length of φ_t is defined as the geodesic distance between φ_0 and φ_1, i.e.*

$$(6.26) \qquad d(\varphi_0, \varphi_1) = \int_0^1 dt \sqrt{\int_M |\dot{\varphi}_t|^2 \eta_{\varphi_t} \wedge (d\eta_{\varphi_t})^n}.$$

Theorem 3. *Let $\mathcal{C} : \varphi(s) : [0, 1] \to \mathcal{H}$ be a smooth path in $\mathcal{H}$, and $\varphi^* \in \mathcal{H}$ be a point. Then, for any s, we have*

$$d(\varphi^*, \varphi(0)) \le d(\varphi^*, \varphi(s)) + d_c(\varphi(0), \varphi(s)),$$

where d_c denotes the length along the curve $\mathcal{C}$. In particular, we have the following triangle inequality

$$d(\varphi^*, \varphi(0)) \le d(\varphi^*, \varphi(1)) + d_c(\varphi(0), \varphi(1)).$$

Furthermore, the space $(\mathcal{H}, d)$ is a metric space. Moreover, the distance function is at least C^1.

The following provides a uniqueness type result for the constant transversal scalar curvature metric (if it exists) in the case $C_1^B(M) \leq 0$.

Theorem 4. *Let (M, ξ, η, Φ, g) be a Sasakian manifold with $C_1^B(M) \leq 0$. Then a constant scalar curvature transverse Kähler metric, if it exists, realizes the global minimum of the $\mathcal{K}$ energy functional in each basic Kähler class. In addition, if either $C_1^B(M) = 0$ or $C_1^B(M) < 0$, then the constant scalar curvature transverse Kähler metric, if it exists, in any basic Kähler class must be unique.*

We would also like to call attention to recent papers [25, 28, 30] on the uniqueness of Sasakian-Einstein metrics and Sasaki-Ricci flow.

7. APPENDIX

We now provide proof of results listed in the previous section following the same argument as in Chen [5], here we make use of our Theorems 1.

Proof of Lemma 10. By the transverse Calabi-Yau theorem in [12], there is a function $\varphi_0 \in \mathcal{H}$, such that

$$(7.1) \qquad \rho_0^T = -\sqrt{-1}\partial_B\bar{\partial}_B \log \det(g_0^T) \leq 0,$$

where ρ_0^T is the transverse Ricci form corresponding to the new Sasakian structure $(\xi, \eta_{\varphi_0}, \Phi_{\varphi_0}, g_{\varphi_0})$. Let $\triangle_0$ be the Laplacian corresponding to the metric g_{φ_0}, and choosing a local coordinates $(x, z^1, \cdots, z^n)$ as in (2.3). Since $V_{\eta_\varphi}(\psi)$ is transverse holomorphic,

$$(7.2) \qquad \begin{aligned} \triangle_0 |V_{\eta_\varphi}(\psi)^\perp|^2_{g_0} &= \nabla d|V_{\eta_\varphi}(\psi)^\perp|^2_{g_0}(\xi, \xi) \\ &\quad + 2g_0^{i\bar{j}}\nabla d|V_{\eta_\varphi}(\psi)^\perp|^2_{g_0}(Y_i, \bar{Y}_j) \\ &= |\nabla^T(V_{\eta_\varphi}(\psi)^\perp)|^2_{g_0} - 2Ric_0^T(V_{\eta_\varphi}(\psi)^\perp, \overline{V_{\eta_\varphi}(\psi)^\perp}) \\ &\geq 0, \end{aligned}$$

where $V_{\eta_\varphi}(\psi)^\perp = V_{\eta_\varphi}(\psi) - \eta_{\varphi_0}(V_{\eta_\varphi}(\psi))\xi$ is the projection of $V_{\eta_\varphi}(\psi)$ to $ker\{\eta_{\varphi_0}\}$, and $Y_i = \frac{\partial}{\partial z^i} - \eta_{\varphi_0}(\frac{\partial}{\partial z^i})\frac{\partial}{\partial x}$ is a basis of $(ker\{\eta_{\varphi_0}\})^{1,0}$. From above inequality, we have $|V_{\eta_\varphi}(\psi)^\perp|^2_{g_0} \equiv constant$. On the other hand, by Remark 3,

$$V_{\eta_\varphi}(\psi)^\perp = h_\varphi^{i\bar{j}} \frac{\partial\psi}{\partial\bar{z}^j} Y_i.$$

If ψ achieve the maximum value at some point P, then $|V_{\eta_\varphi}(\psi)^\perp|^2_{g_0} = 0$ at P. Therefore $d\psi \equiv 0$, that is, $\psi \equiv constant$. $\square$

Proof of Lemma 11. Let $\varphi_t : [a, b] \to \mathcal{H}$ be a smooth path connecting φ' and φ''. Define $\psi(s, t) = s\varphi_t \in \mathcal{H}$, $(s, t) \in [0, 1] \times [a, b]$. Consider

$$\theta = (\frac{\partial\psi}{\partial s}, S_\psi^T - \bar{S})ds + (\frac{\partial\psi}{\partial t}, S_\psi^T - \bar{S})dt,$$

where S_ψ^T is the transverse scalar curvature respect to the Sasakian structure $(\xi, \eta_\psi, \Phi_\psi, g_\psi)$. A direct calculation yields

$$(7.3) \qquad (\frac{\partial\psi}{\partial s}, D_{\frac{\partial\psi}{\partial t}}S_\psi^T)_\psi = (\frac{\partial\psi}{\partial t}, D_{\frac{\partial\psi}{\partial s}}S_\psi^T)_\psi.,$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial\psi}{\partial s}, S_\psi^T - \bar{S})_\psi &= (D_{\frac{\partial\psi}{\partial t}}\frac{\partial\psi}{\partial s}, S_\psi^T - \bar{S})_\psi + (\frac{\partial\psi}{\partial s}, D_{\frac{\partial\psi}{\partial t}}S_\psi^T)_\psi \\
&= (D_{\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial t}, S_\psi^T - \bar{S})_\psi + (\frac{\partial\psi}{\partial t}, D_{\frac{\partial\psi}{\partial s}}S_\psi^T)_\psi \\
&= \frac{\partial}{\partial s}(\frac{\partial\psi}{\partial t}, S_\psi^T - \bar{S})_\psi.
\end{aligned}$$

Therefore, θ is a closed one form on $[0,1]\times[a,b]$. Thus, following the same discussion as in [21], we have:

$$\int_a^b (\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t}dt = \int_0^1 (\varphi, S_{s\varphi}^T - \bar{S})_{s\varphi}ds|_{\varphi=\varphi'}^{\varphi=\varphi''},$$

that is, $\mathcal{M}(\varphi', \varphi'')$ is independent of the path φ_t, and $\mathcal{M}$ satisfies 1-cocycle condition, and it satisfies:

$$\mathcal{M}(\varphi_0, \varphi_1) + \mathcal{M}(\varphi_1, \varphi_0) = 0,$$

and

$$\mathcal{M}(\varphi_0, \varphi_1) + \mathcal{M}(\varphi_1, \varphi_2) + \mathcal{M}(\varphi_2, \varphi_0) = 0.$$

On the other hand, it's easy to check that

$$\mathcal{M}(\varphi, \varphi + C) = 0, \quad \forall \varphi \in \mathcal{H}, C \in \mathbb{R}.$$

From the above 1-cocycle condition,

$$\mathcal{M}(\varphi' + C', \varphi'' + C'') - \mathcal{M}(\varphi', \varphi'') = \mathcal{M}(\varphi'', \varphi'' + C'') - \mathcal{M}(\varphi', \varphi' + C') = 0.$$

The lemma is proved. $\square$

Proof of Lemma 12. Choose a local normal coordinates $(x, z^1, \cdots, z^2)$ as in (2.3) around the point considered. We have

$$\begin{aligned}
(7.4) \qquad D_{\dot\varphi_t}S_t^T &= \frac{\partial}{\partial t}S_t^T - \frac{1}{4}< d_B\dot\varphi_t, d_B S_t^T >_{\varphi_t} \\
&= -\frac{1}{2}(\square_{\varphi_t})^2\dot\varphi_t - \frac{1}{2}< \sqrt{-1}\partial_B\bar\partial_B\dot\varphi_t, \rho_t^T >_{\varphi_t} \\
&\quad -\frac{1}{4}< \partial_B S_t^T, \bar\partial_B\dot\varphi_t >_{\varphi_t} -\frac{1}{4}< \partial_B\dot\varphi_t, \bar\partial_B S_t^T >_{\varphi_t} \\
&= -\frac{1}{2}Re(\{[(h_t^{j\bar{i}}(\dot\varphi_t)_j)_k (h_t)_{s\bar{i}}]_{\bar m} h_t^{k\bar m}\}_{\bar q} h_t^{s\bar q})
\end{aligned}$$

where $(h_t)_{i\bar j} = h_{i\bar j} + \frac{1}{2}(\varphi_t)_{i\bar j}$ and $\square_{\varphi_t} = h_t^{i\bar j}\frac{\partial^2}{\partial z^i \partial\bar z^j}$. From the definition of $\mathcal{K}$-energy,

$$(7.5) \qquad \frac{d}{dt}\mu(\varphi_t) = -(\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t}.$$

Hence

$$\begin{aligned}
(7.6) \qquad \frac{d^2}{dt^2}\mu(\varphi_t) &= -(D_{\dot\varphi_t}\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t} - (\dot\varphi_t, D_{\dot\varphi_t}S_t^T)_{\varphi_t} \\
&= -(D_{\dot\varphi_t}\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t} \\
&\quad + \int_M \frac{1}{2}\dot\varphi_t Re(\{[(h_t^{j\bar{i}}(\dot\varphi_t)_j)_k (h_t)_{s\bar{i}}]_{\bar m} h_t^{k\bar m}\}_{\bar q} h_t^{s\bar q})(d\eta_t)^n \wedge \eta_t \\
&= -(D_{\dot\varphi_t}\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t} \\
&\quad + \int_M \frac{1}{2}(h_t^{j\bar{i}}(\dot\varphi_t)_j)_k (h_t)_{s\bar{i}}(h_t^{s\bar q}(\dot\varphi_t)_{\bar q})_{\bar m} h_t^{k\bar m}(d\eta_t)^n \wedge \eta_t \\
&= -(D_{\dot\varphi_t}\dot\varphi_t, S_t^T - \bar{S})_{\varphi_t} \\
&\quad + \int_M \frac{1}{2}|(h_t^{i\bar j}(\dot\varphi_t)_{\bar j})_{\bar k}d\bar z^k \otimes \frac{\partial}{\partial z^i}|_{\varphi_t}^2 (d\eta_t)^n \wedge \eta_t.
\end{aligned}$$

For any $\varphi_0 \in \mathcal{H}$ and $\psi \in C_B^\infty(M)$, choose a smooth path $\{\varphi_t| -\epsilon \le t \le \epsilon\}$ in $\mathcal{H}$ such that $\dot\varphi_t|_{t=0} = \psi$. The above identity yields

$$\begin{aligned}
(7.7) \qquad (Hess\mu)_{\varphi_0}(\psi, \psi) &= \frac{d^2}{dt^2}\mu(\varphi_t)|_{t=0} - (d\mu)_{\varphi_0}(D_{\dot\varphi_t}\dot\varphi_t|_0) \\
&= \int_M \frac{1}{2}|\bar\partial_B V_{\varphi_t}(\dot\varphi_t)|_{\varphi_t}^2 (d\eta_t)^n \wedge \eta_t \ge 0.
\end{aligned}$$

$\square$

Definition 6. *A smooth path φ_t in the space $\mathcal{H}$ is called an ϵ-approximate geodesic if the following holds:*

$$(7.8) \qquad (\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial \varphi}{\partial t}|^2_{g_\varphi})\eta_\varphi \wedge (d\eta_\varphi)^n = f_\epsilon \eta \wedge (d\eta)^n,$$

where $d\eta_\varphi = d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \phi > 0$ and $0 < f_\epsilon < \epsilon$.

Theorem 1 guarantees the existence of ϵ-approximate geodesic for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$.

Lemma 13. *For any two different points φ_0, φ_1 in $\mathcal{H}$, the geodesic distance between them is positive.*

Proof of Lemma 13. If $\varphi_1 - \varphi_0 \equiv \tilde{C} \neq 0$, where $\tilde{C}$ is a constant. Then, by the definition, $\varphi_t = \varphi_0 + t\tilde{C}$ is the smooth geodesic connecting φ_0 and φ_1. The length of the geodesic is $|\tilde{C}|$, i.e. $d(\varphi_0, \varphi_1) = \tilde{C} > 0$. Therefore, we may assume that $\varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0))$ is not identically zero. Let $\iota(t) = (\varphi_0 + t(\varphi_1 - \varphi_0)) - (\varphi_0)$, $t \in [0, 1]$. We compute that

$$\iota'(t) = \int_M (\varphi_1 - \varphi_0)\, d\nu(\varphi_0 + t(\varphi_1 - \varphi_0)),$$

and

$$\iota''(t) = -\int_M \frac{1}{4}|d_B(\varphi_1 - \varphi_0)|^2_{g_t}\, d\nu(\varphi_0 + t(\varphi_1 - \varphi_0)) \leq 0,$$

where $d\nu(\varphi) = \eta_\varphi \wedge (d\eta_\varphi)^n$. In turn, $\iota'(1) \leq \iota(1) - \iota(0) \leq \iota'(0)$. That is

$$(7.9) \qquad \int_M \varphi_1 - \varphi_0\, d\nu(\varphi_1) \leq \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0) \leq \int_M \varphi_1 - \varphi_0\, d\nu(\varphi_0).$$

This means that the function $\varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0))$ must take both positive and negative values.

Let $\tilde{\varphi}_t$ is a ϵ-approximate geodesic between φ_0 and φ_1. From the estimates in the previous sections, we can suppose that $\max_{M \times [0,1]} |\varphi'(t)|$ have an uniform bound independent on ϵ. Since $\tilde{\varphi}''_t > 0$,

$$(7.10) \qquad \tilde{\varphi}'(0) \leq \varphi_1 - \varphi_0 \leq \tilde{\varphi}'(1).$$

Let $E_\epsilon(t) = \int_M (\tilde{\varphi}')^2 d\nu_{\tilde{\varphi}_t}$ for any $t \in [0, 1]$. If $\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0) \geq 0$, set $t = 1$, by (7.10),

$$(7.11) \qquad \begin{aligned} \sqrt{E_\epsilon(1)} &\geq \int_M |\tilde{\varphi}'(1)| d\nu_{\tilde{\varphi}_1} \\ &\geq \int_{\tilde{\varphi}'(1) > \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} \tilde{\varphi}'(1) d\nu_{\varphi_1} \\ &\geq \int_{\varphi_1 - \varphi_0 > \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} (\varphi_1 - \varphi_0) d\nu_{\varphi_1} \\ &\geq \int_{\varphi_1 - \varphi_0 > \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} (\varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0))) d\nu_{\varphi_1} > 0. \end{aligned}$$

If $\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0) \leq 0$, the similar argument yields,

$$(7.12) \sqrt{E_\epsilon(0)} \leq -\int_{\varphi_1 - \varphi_0 < \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} \varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)) d\nu_{\varphi_0} > 0.$$

On the other hand, since $\tilde{\varphi}_t$ is an ϵ approximate geodesic, it's easy to check that

$$(7.13) \qquad |\frac{d}{dt}E_\epsilon(t)| \leq C\epsilon,$$

where C is a uniform constant. This implies

$$|E_\epsilon(t_1) - E_\epsilon(t_2)| \leq C\epsilon$$

for any $t_1, t_2 \in [0, 1]$. Thus

(7.14) $\sqrt{E_\epsilon(t)} \geq e - C\epsilon,$

where $e = \min\{-\int_{\varphi_1 - \varphi_0 < \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} \pi d\nu_{\varphi_0}, \int_{\varphi_1 - \varphi_0 > \mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0)} \pi d\nu_{\varphi_1}\} > 0$, and $\pi = \varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0))$. Therefore,

(7.15) $\begin{aligned} d(\varphi_0, \varphi_1) &= \lim_{\epsilon \to 0} \int_0^1 \sqrt{E_\epsilon(t)} \, dt \\ &\geq e > 0. \end{aligned}$

□

Lemma 14. *Let $\varphi_i(s) : [0, 1] \to \mathcal{H}$ $(i = 0, 1)$ are two smooth curves in $\mathcal{H}$. For any $0 < \epsilon \leq 1$, there exist two parameter families of smooth curves $\mathcal{C}(t, s, \epsilon) : \varphi(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, 1] \to \mathcal{H}$ such that the following properties hold:*

(1) *Let $\psi_{s,\epsilon}(r, \cdot) = \varphi(2(r - 1), s, \epsilon) + 4 \log r \in C^\infty(\overline{M})$ solving*

$$(\Omega_\psi)^{n+1} = \epsilon \bar{\omega}^{n+1}$$

with boundary conditions: $\psi_{s,\epsilon}(1, \cdot) = \varphi_0(s, \cdot)$ and $\psi_{s,\epsilon}(\frac{3}{2}, \cdot) = \varphi_1(s, \cdot)$, and $\Omega_\psi > 0$.

(2) *There exists a uniform constant C which depends only on φ_0 and φ_1 such that*

$$|\varphi| + |\frac{\partial \varphi}{\partial t}| + |\frac{\partial \varphi}{\partial s}| \leq C; \quad 0 < \frac{\partial^2 \varphi}{\partial t^2} \leq C; \quad \frac{\partial^2 \varphi}{\partial s^2} \leq C.$$

(3) *For fixed s, let $\epsilon \to 0$, the curve $\mathcal{C}(s, \epsilon)$ converge to the unique weak geodesic connecting $\varphi_0(s)$ and $\varphi_1(s)$ in the weak $C^{1,1}$ topology.*

(4) *Define the energy element along $\varphi(t, s, \epsilon) \in \mathcal{H}$ as*

$$E(t, s, \epsilon) = \int_M |\frac{\partial \varphi}{\partial t}|^2 d\nu_{\varphi(t,s,\epsilon)}$$

where $d\nu_\varphi = \eta_\varphi \wedge (d\eta_\varphi)^n$. There exist a uniform constant C which independent of ϵ, such that

$$|\frac{\partial E}{\partial t}| \leq C\epsilon.$$

Proof of Lemma 14. Everything follows from Theorem 1, except $|\frac{\partial \varphi}{\partial s}| \leq C$ and $\frac{\partial^2 \varphi}{\partial s^2} \leq C$. The inequalities above follow from the maximum principle directly since

(7.16) $\tilde{g}^{\alpha\bar{\beta}}[(\frac{\partial \psi}{\partial s})_{\alpha\bar{\beta}} - r_{\alpha\bar{\beta}} \frac{\partial}{\partial r}(\frac{\partial \psi}{\partial s})] = 0,$

and

(7.17) $\tilde{g}^{\alpha\bar{\beta}}[(\frac{\partial^2 \psi}{\partial s})_{\alpha\bar{\beta}} - r_{\alpha\bar{\beta}} \frac{\partial}{\partial r}(\frac{\partial^2 \psi}{\partial s^2})] \geq 0,$

where $\tilde{g}$ is the Hermitian metric induced by the positive $(1, 1)$-form Ω_ψ. □

Proof of Theorem 3. For any $\epsilon > 0$, by Lemma 14 there exist two parameter families of smooth curves $\mathcal{C}(t, s, \epsilon) : \tilde{\varphi}(t, s, \epsilon) \in \mathcal{H}$ such that it satisfies $(\Omega_\psi)^{n+1} = \epsilon \bar{\omega}^n$ or equivalently

$$(\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial \tilde{\varphi}}{\partial t}|^2_{g_{\tilde{\varphi}}})\eta_{\tilde{\varphi}} \wedge (d\eta_{\tilde{\varphi}})^n = (\frac{t}{2} + 1)^{-2}\epsilon\eta \wedge (d\eta)^n,$$

$\tilde{\varphi}(0, s, \epsilon) = \varphi^*$ and $\tilde{\varphi}(1, s, \epsilon) = \varphi(s)$. For each s fixed, denote the length of curve $\tilde{\varphi}(t, s, \epsilon)$ form φ^* to $\varphi(s)$ by $L(s, \epsilon)$, and denote the length from $\varphi(0)$ to $\varphi(s)$ along curve $\mathcal{C}$ by $l(s)$. In what follows, we assume that energy element $E > 0$ (we may replace $\sqrt{E}$ by $\sqrt{E + \delta^2}$ and let $\delta \to 0$). We compute

$$
(7.18) \quad
\begin{aligned}
\frac{dL(s,\epsilon)}{ds} &= \int_0^1 \frac{dt}{\sqrt{E(t,s,\epsilon)}} (D_{\frac{\partial\tilde{\varphi}}{\partial s}} \frac{\partial\tilde{\varphi}}{\partial t}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}} \\
&= \int_0^1 \frac{dt}{\sqrt{E(t,s,\epsilon)}} (D_{\frac{\partial\tilde{\varphi}}{\partial t}} \frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}} \\
&= \int_0^1 \frac{1}{\sqrt{E(t,s,\epsilon)}} [\frac{d}{dt}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}} - (D_{\frac{\partial\tilde{\varphi}}{\partial t}} \frac{\partial\tilde{\varphi}}{\partial t}, \frac{\partial\tilde{\varphi}}{\partial s})_{\tilde{\varphi}}] \\
&= [\frac{1}{\sqrt{E(t,s,\epsilon)}}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}}]|_0^1 - \int_0^1 \frac{dt}{\sqrt{E(t,s,\epsilon)}} (D_{\frac{\partial\tilde{\varphi}}{\partial t}} \frac{\partial\tilde{\varphi}}{\partial t}, \frac{\partial\tilde{\varphi}}{\partial s})_{\tilde{\varphi}} \\
&\quad + \int_0^1 [E^{-\frac{3}{2}}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}} (D_{\frac{\partial\tilde{\varphi}}{\partial t}} \frac{\partial\tilde{\varphi}}{\partial t}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}}] \, dt \\
&\geq \frac{1}{\sqrt{E(1,s,\epsilon)}}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}}|_{t=1} - C\epsilon,
\end{aligned}
$$

and

$$
\frac{dl(s)}{ds} = \sqrt{(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s})_\varphi} = \sqrt{(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial s})_{\tilde{\varphi}}|_{t=1}}.
$$

Set $F(s, \epsilon) = L(s, \epsilon) + l(s)$. By the Schwartz inequality, $\frac{dF(s,\epsilon)}{ds} \geq -C\epsilon$. In turn, $F(s, \epsilon) - F(0, \epsilon) \geq -C\epsilon$. Letting $\epsilon \to 0$,

$$
d(\varphi^*, \varphi(0)) \leq d(\varphi^*, \varphi(s)) + d_c(\varphi(0), \varphi(s)).
$$

The triangle inequality in the Theorem can be deduced from the above inequality by choosing appropriate ϵ-approximate geodesics.

We now verify the second part of the theorem. By taking $\varphi^* = \varphi(1)$ in the triangle inequality, we know that the geodesic distance is no greater than the length of any curve connecting the two end points. Then, Lemma 13 implies that $(\mathcal{H}, d)$ is a metric space. We only need to show the differentiability of the distance function. Suppose $\varphi^* \neq \varphi(s_0)$, from (7.18), we have

$$
|\frac{dL(s, \epsilon)}{ds} - \frac{1}{\sqrt{E(1, s, \epsilon)}}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}}|_{t=1}| \leq C\epsilon
$$

Let $\epsilon \to 0$, it follows that

$$
\frac{d}{ds} d(\varphi^*, \varphi(s))|_{s=s_0} = \lim_{\epsilon \to 0} \frac{1}{\sqrt{E(1, s_0, \epsilon)}}(\frac{\partial\tilde{\varphi}}{\partial s}, \frac{\partial\tilde{\varphi}}{\partial t})_{\tilde{\varphi}}|_{t=1, s=s_0}.
$$

□

If $C_1^B(M) \leq 0$, by the transverse Calabi-Yau theorem in [12], there exists $\tilde{\varphi} \in \mathcal{H}$ such that the transverse Ricci curvature $\tilde{Ric}^T$ of the tranverse Kähler metric $g_{\tilde{\varphi}}^T$ is nonpositive, where $g_{\tilde{\varphi}}^T$ is induced by the Sasakian structure $(\xi, \eta_{\tilde{\varphi}}, \Phi_{\tilde{\varphi}}, g_{\tilde{\varphi}})$. One may assume that $\tilde{Ric}^T < 0$ if $C_1^B(M) < 0$; and $\tilde{Ric}^T \equiv 0$ if $C_1^B(M) = 0$. We will take $\eta_{\tilde{\varphi}}$ as the background contact form, we write η for $\eta_{\tilde{\varphi}}$.

For any two point $\varphi_0, \varphi_1 \in \mathcal{H}$, by Theorem 1, there is an ϵ-approximate geodesic $\varphi(t)$ satisfies (1.12). We have

$$
(7.19) \quad \rho_\varphi^T - \rho^T = \sqrt{-1}\partial_B \bar{\partial}_B \log Q,
$$

where ρ^T_φ and ρ^T are the transverse Ricci forms of g^T_φ and g^T respectively, and $Q = \varphi'' - \frac{1}{4}|d_B\varphi'|^2_{g_\varphi}$. Then,

$$
\begin{aligned}
(7.20) \quad
&\int_M S^T_t Q \eta_\varphi \wedge (\eta_\varphi)^n = \int_M 2nQ\rho^T_\varphi \wedge (\eta_\varphi)^{n-1} \wedge \eta_\varphi \\
&= \int_M 2nQ\sqrt{-1}\partial_B\bar\partial_B \log Q \wedge (\eta_\varphi)^{n-1} \wedge \eta_\varphi \\
&+ \int_M 2nQ\tilde\rho^T \wedge (\eta_\varphi)^{n-1} \wedge \eta_\varphi \\
&= -\int_M |\partial_B Q|^2_\varphi Q^{-1}(\eta_\varphi)^n \wedge \eta_\varphi + \int_M Q tr_{g_\varphi}(\tilde{Ric}^T)(\eta_\varphi)^n \wedge \eta_\varphi.
\end{aligned}
$$

Consider the $\mathcal{K}$ energy map μ on $\mathcal{H}$, we have

$$
(7.21) \qquad \frac{d}{dt}\mu(\varphi(t)) = -(\varphi', S^T_t - \bar S)_{\varphi(t)}.
$$

By Lemma 12 and (7.19),

$$
\begin{aligned}
(7.22) \quad
\frac{d^2}{dt^2}\mu(\varphi(t)) &= -(D_{\varphi'}\varphi', S^T_t - \bar S)_\varphi + \int_M \tfrac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|^2_\varphi (d\eta_t)^n \wedge \eta_t \\
&= \int_M \tfrac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|^2_\varphi (d\eta_t)^n \wedge \eta_t + \epsilon\bar S \\
&+ \int_M |\partial_B Q|^2_\varphi Q^{-1}(\eta_\varphi)^n \wedge \eta_\varphi - \int_M Q tr_{g_\varphi}(\tilde{Ric}^T)(\eta_\varphi)^n \wedge \eta_\varphi.
\end{aligned}
$$

Proof of Theorem 4. Let $\mathcal{K}$ be the space of all transverse Kähler metrics in the same basic Kähler class, we know $\mathcal{K} \cong \mathcal{H}_0 \subset \mathcal{H}$. Suppose $\varphi_0 \in \mathcal{H}$ satisfy $S^T_{\varphi_0} \equiv constant$. For any point $\varphi_1 \in \mathcal{H}$, let $\varphi(t)$ be an ϵ-approximate geodesic as defined in (1.12). Since Ric^T is nonpositive by the assumption, (7.22) implies,

$$
(7.23) \qquad \frac{d^2}{dt^2}\mu(\varphi(t)) > -\epsilon C,
$$

where C is an uniform constant. On the other hand, since $S^T_{\varphi_0} \equiv constant$, $\frac{d}{dt}\mu(\varphi(t))|_{t=0} = 0$. Hence

$$
(7.24) \qquad \mu(\varphi(t)) - \mu(\varphi(0)) \geq -\epsilon C \frac{t^2}{2}.
$$

Let $t = 1$ and $\epsilon \to 0$, we have $\mu(\varphi_1) \geq \mu(\varphi_0)$. The first part of the theorem is proved since φ_1 is arbitrary.

Let φ_0 and φ_1 be two constant scalar curvature transverse Kähler metrics in the same basic Kähler class $\mathcal{K}$. By the identity between $\mathcal{K}$ and $\mathcal{H}_0 \subset \mathcal{H}$, we can consider φ_0 and φ_1 as two functions in $\mathcal{H}$. Let $\{\varphi(t)|t \in [0,1]\}$ be a ϵ-approximate geodesic in $\mathcal{H}$ and satisfies (1.12). Integrating (7.22) from $t = 0$ to $t = 1$,

$$
\begin{aligned}
(7.25) \quad
(\tfrac{d}{dt}\mu(\varphi(t)))|^1_{t=0} &= \int^1_0 \int_M \tfrac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|^2_\varphi (d\eta_t)^n \wedge \eta_t \, dt + \epsilon\bar S \\
&+ \int^1_0 \int_M |\partial_B Q|^2_\varphi Q^{-1}(\eta_\varphi)^n \wedge \eta_\varphi \, dt - \int^1_0 \int_M Q tr_{g_\varphi}(Ric^T)(\eta_\varphi)^n \wedge \eta_\varphi \, dt.
\end{aligned}
$$

Since φ_0 and φ_1 are two metrics with transverse constant scalar curvature, by (7.5), $(\frac{d}{dt}\mu(\varphi(t)))|^1_{t=0} = 0$. (1.12) and (7.25) imply

$$
\begin{aligned}
(7.26) \quad
&\int^1_0 \int_M \{\tfrac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|^2_\varphi Q^{-1} + |\partial_B \log Q|^2_\varphi\}(d\eta)^n \wedge \eta \, dt \\
&= \int^1_0 \int_M tr_{g_\varphi}(Ric^T)(d\eta)^n \wedge \eta \, dt - \bar S.
\end{aligned}
$$

If $C^B_1(M) = 0$, then the constant $\bar S = 0$, by the initial assumption, $Ric^T = 0$. Consequently

$$
(7.27) \qquad \int^1_0 \int_M \{\frac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|^2_\varphi Q^{-1} + |\partial_B \log Q|^2_\varphi\}(d\eta)^n \wedge \eta \, dt = 0
$$

This implies the Hamiltonian vector field $V_\varphi(\varphi')$ is transversal holomorphic. By Lemma 10, $\varphi'(t)$ is constant for each t. Therefore φ_0 and φ_1 represent the same transverse Kähler metric. That is, there exists at most one constant scalar curvature transverse Kähler metric in each basic Kähler class when $C_1^B(M) = 0$.

If $C_1^B(M) < 0$, then $Ric^T < -cg^T$ for some positive constant c. By (7.26), we have

$$(7.28) \quad \begin{aligned} &\int_0^1 \int_M \{\tfrac{1}{2}|\bar\partial_B V_{\eta_\varphi}(\varphi')|_\varphi^2 Q^{-1} + |\partial_B \log Q|_\varphi^2\}(d\eta)^n \wedge \eta\, dt \\ &\leq -c\int_0^1 \int_M tr_{g_\varphi}(g^T)(d\eta)^n \wedge \eta\, dt - \bar S.\end{aligned}$$

where $\bar S$ is a negative constant depending only the basic Kähler class. Following the same discussion in [5] (section 6.2), we may argue $\bar\partial_B V_\varphi(\varphi') = 0$ in some weak sense. The following is a sketch of proof.

From the estimates in Theorem 1 and (5.10), there exist an uniform positive constant C which independent on ϵ, such that $Q \leq \varphi'' \leq C$. In what follows, we will denote C as an uniform constant under control, and set

$$(7.29) \quad d\tilde\nu = (d\eta)^n \wedge \eta, \quad X = V_{\eta_\varphi}(\varphi') - \eta(V_{\eta_\varphi}(\varphi'))\xi.$$

First we have an integral estimate on $Q^{\frac{q}{2-q}}$ $(1 < q < 2)$ with respect to the measure $d\nu dt$:

$$(7.30) \quad \begin{aligned} \int_{M\times[0,1]} Q^{\frac{q}{2-q}} d\nu dt &\leq C\int_{M\times[0,1]} Q^{\frac{1}{n}} d\nu dt \\ &= C\int_{M\times[0,1]} \{Q\frac{\det g_\varphi^T}{\det g^T}\}^{\frac{1}{n}} \cdot \{\frac{\det g_\varphi^T}{\det g^T}\}^{-\frac{1}{n}} d\nu dt \\ &\leq C\epsilon^{\frac{1}{n}} \int_{M\times[0,1]} tr_{g_\varphi}(g^T) d\nu dt \to 0.\end{aligned}$$

The following inequality shows that vector field X is uniformly bounded in L^2 with respect to the measure $d\nu dt$.

$$(7.31) \quad \begin{aligned} \int_{M\times[0,1]} |X|_g^2 d\nu dt &= \int_{M\times[0,1]} g_{\alpha\bar\beta} g_\varphi^{\alpha\bar\delta} g_\varphi^{\gamma\bar\gamma}(\varphi')_\gamma(\varphi')_{\bar\delta} d\nu dt \\ &\leq \int_{M\times[0,1]} tr_{g_\varphi}(g^T)|d_B\varphi'|_\varphi^2 d\nu dt \leq C.\end{aligned}$$

A direct calculation yields

$$(7.32) \quad \begin{aligned} \int_{M\times[0,1]} |\bar\partial_B V_\varphi(\varphi')|_\varphi^q d\nu dt &= \int_{M\times[0,1]} |\bar\partial_B V_\varphi(\varphi')|_\varphi^q Q^{-\frac{q}{2}} Q^{\frac{q}{2}} d\nu dt \\ &\leq \{\int_{M\times[0,1]} |\bar\partial_B V_\varphi(\varphi')|_\varphi^2 Q^{-1} d\nu dt\}^{\frac{q}{2}} \{\int_{M\times[0,1]} Q^{\frac{q}{2-q}} d\nu dt\}^{\frac{2-q}{2}} \\ &\leq C\{\int_{M\times[0,1]} Q^{\frac{q}{2-q}} d\nu dt\}^{\frac{2-q}{2}} \to 0.\end{aligned}$$

Therefore, $|\bar\partial_B V_\varphi(\varphi')|$ can be viewed as a function in $L^2(M\times[0,1])$. It has a weak limit in L^2 and it's L^q $(1 < q < 2)$ norm tends to 0 as $\epsilon \to 0$.

As above, let $\mathcal{D} = ker\{\eta\}$ be the contact sub-bundle with respect to the Sasakian structure (ξ, η, Φ, g). Let $Y \in \Gamma(\mathcal{D}^{1,0})$. Choose local coordinates $(x, z^1, \cdots, z^2)$ on the Sasakian manifold M. For $Y = Y^i(\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\xi)$, define $\bar\partial_B Y \in \Gamma(\wedge_B^{0,1}(M) \otimes \mathcal{D}^{1,0})$ by $\frac{\partial Y^i}{\partial \bar z^j} d\bar z^j \otimes (\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\xi)$. One may check that

$$(7.33) \quad |\bar\partial_B X|_g \leq C\sqrt{tr_{g_\varphi}g}|\bar\partial_B V_{\eta_\varphi}\varphi'|_{g_\varphi},$$

$$(7.34) \quad \begin{aligned} \int_{M\times[0,1]} |d_B \log \frac{\det g_\varphi^T}{\det g^T}|_g^2 d\nu dt &= \int_{M\times[0,1]} |d_B \log Q|_g^2 d\nu dt \\ &\leq C\int_{M\times[0,1]} |d_B \log Q|_{g_\varphi}^2 d\nu dt \leq C,\end{aligned}$$

and

$$(7.35) \qquad \int_{M \times [0,1]} (\frac{\det g^T}{\det g_\varphi^T})^{\frac{1}{n}} d\nu dt \leq \int_{M \times [0,1]} tr_{g_\varphi} g^T d\nu dt \leq C.$$

By the C_w^2 estimate in theorem 1, there is $c > 0$ such that $e^{-c} \frac{\det g_\varphi^T}{\det g^T} \leq 1$. Now define a vector field Y by

$$(7.36) \qquad Y = X e^{-c} \frac{\det g_\varphi^T}{\det g^T}.$$

We have

$$(7.37) \qquad |Y|_g = |X|_g \frac{\det g_\varphi^T}{\det g^T} \leq C.$$

and

$$
\begin{aligned}
(7.38) \qquad & \int_{M \times [0,1]} |\bar\partial_B Y - \bar\partial_B (\log \frac{\det g_\varphi^T}{\det g^T}) \otimes Y|_g^q d\nu dt \\
= \quad & \int_{M \times [0,1]} (|\bar\partial_B X|_g \frac{\det g_\varphi^T}{\det g^T})^q d\nu dt \\
= \quad & \int_{M \times [0,1]} (\sqrt{tr_{g_\varphi} g} \frac{\det g_\varphi^T}{\det g^T})^q |\bar\partial_B V_{\eta_\varphi} \varphi'|_{g_\varphi}^q d\nu dt \\
= \quad & C \int_{M \times [0,1]} |\bar\partial_B V_{\eta_\varphi} \varphi'|_{g_\varphi}^q d\nu dt \to 0
\end{aligned}
$$

for any $0 < q < 2$.

Note that X, Y, $\bar\partial_B Y$ and $\frac{\det g_\varphi^T}{\det g^T}$ are geometric quantities which depend on ϵ, and their respect Sobolev norms are uniformly bounded. We have $X(\epsilon) \rightharpoonup X$ weakly in $L^2(M \times [0,1])$, $Y(\epsilon) \rightharpoonup Y$ weakly in $L^\infty(M \times [0,1])$ and $\frac{\det g_\varphi^T}{\det g^T}(\epsilon) \rightharpoonup u$ weakly in $L^\infty(M \times [0,1])$, as $\epsilon \to 0$. Furthermore, in local coordinates $(x, z^1, \cdots, z^n)$, since $\nabla_\xi^T X(\epsilon) \equiv 0$ and $\xi(\frac{det g_\varphi^T}{det g^T}(\epsilon)) \equiv 0$ for any ϵ, then functions v, X^i and Y^i are all independent of x, where $X = X^i(\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\xi)$ and $Y = Y^i(\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\xi)$.

Let $v = -\log u$. With the choice of c in the definition of Y in (7.36), $v \geq 0$ and it satisfies the following two equations

$$(7.39) \qquad \bar\partial_B Y + \bar\partial_B v \otimes Y = 0, \quad and \quad Y = X e^{-v}$$

in the sense of L^q for any $1 < q < 2$. From (7.31), (7.34) and (7.35), we have the following estimates

$$\int_{M \times [0,1]} |X|_g^2 + e^{\frac{v}{n}} + |\bar\partial_B v|_g^2 d\nu dt \leq C.$$

Define a new sequence of vector fields $X_k = Y \sum_{i=0}^k \frac{v^i}{i!}$. This is well defined since $v \in L^p(M \times [0,1])$ for any $p > 1$. It's easy to check that:

$$
(7.40) \qquad
\begin{aligned}
\|X_k\|_{L^2(M \times [0,1])}^2 + \|X_m - X_k\|_{L^2(M \times [0,1])}^2 & \leq & \|X_m\|_{L^2(M \times [0,1])}^2 \\
& \leq & \|X\|_{L^2(M \times [0,1])}^2,
\end{aligned}
$$

where $k < m$. Thus, X_k is a Cauchy sequence in $L^2(M \times [0,1])$ and there exists a strong limit X_∞ in $L^2(M \times [0,1])$. By definition, one may check that $X_\infty = X$ in the sense of L^q for any $1 < q < 2$. In local coordinates $(x, z^1, \cdots, z^n)$ as in (2.3) in an open set U, the functions X_∞^i are all invariant in x direction, where

$X_\infty = X_\infty^i(\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i})\xi)$. For any vector valued smooth function $\theta = (\theta^1, \cdots, \theta^n)$ supported in $U \times [0,1]$, and any $1 \le j \le n$, we have

$$
\begin{aligned}
|\int_{U\times[0,1]} \sum_{i=1}^n X_\infty^i \frac{\partial}{\partial \bar{z}^j}(\bar{\theta}^i)| &= \lim_{k\to\infty} |\int_{U\times[0,1]} \sum_{i=1}^n X_k^i \frac{\partial}{\partial \bar{z}^j}(\bar{\theta}^i)| \\
&= \lim_{k\to\infty} |\int_{U\times[0,1]} \sum_{i=1}^n (X_k^i - X_{k-1}^i)\frac{\partial v}{\partial \bar{z}^j}\bar{\theta}^i| \\
&\le \lim_{k\to\infty} C\|X_k - X_{k-1}\|_{L^2} = 0.
\end{aligned}
$$

The above implies that component functions X_∞^i are weak holomorphic and x-invariant. That is X_∞ is a weak transverse holomorphic vector field for almost all $t \in [0,1]$. Recall that $\|X_\infty\|_{L^2(M\times[0,1])} \le C$. This implies that X_∞ is in $L^2(M)$ for almost all $t \in [0,1]$. Therefore X_∞ must be transverse holomorphic for almost all $t \in [0,1]$. Since $Ric^T < -cg^T$ for some positive constant c, by (7.2) in lemma 10, $X_\infty(t) \equiv 0$ for those t where $X_\infty(t)$ is transverse holomorphic. Thus $X \equiv 0$. We conclude that φ' is constant for each t fixed. Therefore, φ_0 and φ_1 differ only by a constant, and they represent the same transverse Kähler metric. $\square$

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Department of Mathematics and Statistics, McGill University, Canada
E-mail address: guan@math.mcgill.ca

Department of Mathematics, Zhejiang University, P. R. China
E-mail address: xizhang@zju.edu.cn